Global Financial Management

Bond Valuation

Latest Revision: August 23, 1999

2.0 Bonds

Bonds are securities that establish a creditor relationship between the purchaser (creditor) and the issuer (debtor). The issuer receives a certain amount of money in return for the bond, and is obligated to repay the principal at the end of the lifetime of the bond (maturity). Typically, bonds also require coupon or interest payments. Since all these payments are determined as part of the contracts, bonds are also called fixed income securities.

A straight bond is one where the purchaser pays a fixed amount of money to buy the bond. At regular periods, she receives an interest payment, called the *coupon payment*. The final interest payment and the principal are paid at a specific date of maturity. Bonds usually pay a standard *coupon* amount, C, at regular intervals and this represents the interest on the bond. At the maturity of the bond, the final interest payment is made plus the principal amount (or *par* amount) is repaid.

Some bonds do not make a coupon payment. These bonds are bought for less than their face value (we say such bonds are bought at a *discount*). Bonds that do not pay coupons are often called *Zero Coupon Bonds*.

2.1 Objectives

At the end of this lecture you should be able to:

- Value a straight bond and a zero-coupon bond using present discounted value techniques
- Understand the relationship between interest rates and bond prices
- Understand the bond reporting conventions and determine the actual price of a bond from the reported figures.
- Determine the yield to maturity for a straight bond
- Understand the relationships between zero coupon bonds and coupon bonds
- Analyze bond price dynamics and predict how bond prices respond to changes in interest rates
- Explain why coupon bonds and zero coupon bonds react differently to changes in interest rates.
- Explain the relationship between real and nominal interest rates.
- Explain and apply the concept of a forward rate

2.2 Issuers of Bonds

Bonds are issued by many different entities, including corporations, governments and government agencies. We will consider two major types of issuers: The United States Treasury and U.S. Corporations.

2.2.1 Treasuries

There are three major types of treasury issues:

- **Treasury Bills**. T-bills have maturities of up to 12 months. They are zero coupon bonds, so the only cash flow is the face value received at maturity.
- **Treasury Notes**. Notes have maturities between one year and ten years. They are straight bonds and pay coupons twice per year, with the principal paid in full at maturity.

- **Treasury Bonds**. T-Bonds may be issued with any maturity, but usually have maturities of ten years or more. They are straight bonds and pay coupons twice per year, with the principal paid in full at maturity.

U.S. Treasury bonds and notes pay interest semi-annually, (e.g., in May and November). A bond with a quoted annual coupon of 8.5% really makes coupon payments of $8.5/2 or $4.25 per $100 of bond value twice a year.

Treasury securities are debt obligations of the United States government, issued by the treasury department. They are backed by the full faith and credit of the U.S. government and its taxing power. They are considered to be free of default risk.

2.2.2 Corporate Bonds

We will consider three major types of corporate bonds:

- **Mortgage Bonds.** These bonds are secured by real property such as real estate or buildings. In the event of default, the property can be sold and the bondholders repaid.

- **Debentures.** These are the normal types of bonds. It is unsecured debt, backed only by the name and goodwill of the corporation. In the event of the liquidation of the corporation, holders of debentures are repaid before stockholders, but after holders of mortgage bonds.

- **Convertible Bonds.** These are bonds that can be exchanged for stock in the corporation.

In the United States, most corporate bonds pay two coupon payments per year until the bond matures, when the principal payment is made with the last coupon payment.

2.3 Analysis of bond prices

We will use the following notation:

B	Market price of the bond
F	Principal payment (Face or par value)
C	Annual coupon rate of the bond
m	The number of coupon payments per year
c	Periodic coupon rate (*C/m*)
R	APR (Annual Percentage Rate) for today's cash flows
i	Effective periodic interest rate (*i=R/m*)
t	The number of years to maturity.
N	The total number of periods (*Note: N = mt*)

Example 1

Suppose a zero coupon bond with par value of $100 is trading for $80. It matures in six years from now. The annual percentage rate is 7%. Then, in terms of our notation:

B = $80, F = 100, t=6 and R=7%.

Example 2

Suppose a 20% coupon bond with par value of $100 is trading for $110. It matures in three years from now and pays the coupon semi-annually. The annual percentage rate is 13%. Then, in terms of our notation:

B = $110, F = 100, C = 20%, c=10%, m = 2, t=3, N=6, R=13% and i = 6.5%.

We can use the tools that we have developed to calculate present value and future value to examine zero coupon bonds. A *zero coupon bond* is a bond that pays a fixed par amount at maturity *t* and no coupons prior to this period. For simplicity, we will assume that the par value is $1. They are traded in the U.S. with names like *zeros*, *money multipliers*, *CATs*, *TIGRs*, and *STRIPs*. CATs are Salomon Bros' *Certificates of Accrual on Treasury Securities*. TIGRs are Merrill Lynch's *Treasury Investment Growth and Receipts* and STRIPS are *Separate Trading of Registered Interest and Principal of Securities*.

These securities sell at a substantial discount from their *par* value of $1. The discount represents the interest earned on the investment through its life.

Example 3

As an example of a bond price schedule, consider the quotations for CATs (Certificates of Accrual on Treasury Bills) that are drawn from the *Wall Street Journal*.

Maturity	Price
3 years	74.63
5 years	52.00
12 years	41.00
16 years	32.00
17 years	28.00
19 years	18.75

Note that the bond prices decline with time.

2.4 Bond Prices

We can link the level of the Interest rate 'R' to the price of a zero coupon bond B_0. Writing out the formula for the price of the bond we have:

$$B_0 = \frac{F}{(1 + R/m)^N} \quad (1)$$

The immediate consequences are:

Higher interest rates imply lower zero coupon bond prices.

Using Zeros to Value bonds with coupons

Consider the 3-year coupon bond from example 2. The cashflows from this bond are:

CF_1	10
CF_2	10
CF_3	10
CF_4	10
CF_5	10
CF_6	110

In addition, the annual interest rate is equal to 13% and hence the semi-annual rate is 6.5%. What is the value of this bond?

To answer this question, we can think of the cash flows as a portfolio of zero coupon bonds that are mature every six months for the next three years. We can construct a *replicating* portfolio by purchasing zeros with $1 par price. This portfolio will generate the same cashflows that would be earned if we held the coupon bond.

Period	**Zeros' months to Maturity**	**Cash Flow**	**# of Zeros**	**Price of zeros with par of $1**
1	6	10	10	0.939
2	12	10	10	0.882
3	18	10	10	0.828
4	24	10	10	0.777
5	30	10	10	0.730
6	36	110	110	0.685

Suppose that there existed prices for the zero coupon bonds for every maturity we are concerned with. We can then exactly replicate the cash flows with zero coupon bonds:

Example 4

Valuing Cash Flows from Zero Coupon Prices

Zero Coupon Payoffs							
period	**10 6 months**	**10 1 year**	**10 18months**	**10 2 years**	**10 30months**	**110 3 years**	**Entire Portfolio**
1	10	0	0	0	0	0	10
2	0	10	0	0	0	0	10
3	0	0	10	0	0	0	10
4	0	0	0	10	0	0	10
5	0	0	0	0	10	0	10
6	0	0	0	0	0	110	110
Replication Cost	**9.39**	**8.82**	**8.28**	**7.77**	**7.30**	**75.39**	**116.95**

The cost of replicating the cash flows of the bond is $116.95. You can go into the market and buy zero coupon bonds that will give you the same payoffs at the same dates as your bond. This is a way of looking at the present value of the stream of coupons and face value. Previously, we considered one cash flow at period *n* and we derived a method to bring it back to today. Clearly, we can think of that method as buying zeros today that replicate that cash flow.

2.5 No Arbitrage

We can value cash flows by creating an alternative portfolio of traded assets, which exactly replicates the cash flows of the bond. By the principle of no arbitrage, the replicating portfolio must have the same value as the bond. The principal of arbitrage says that no money can be made for free. Equal cash flows in all the different states of the world must have the same value.

Suppose that this was not true. Then if a trader has a higher value, then she would sell the higher priced asset, and buy the lower priced asset with the proceeds. Since they have the same cash flows, the investors would make a profit without investing any money. The lower value investment would have selling pressure as investors dump it and go to the higher value investment. The higher valued investment would be bid up as investors try to take advantage of the arbitrage opportunity. The prices equalize as a result.

Example 5: Arbitrage

Reconsider the three-year coupon bond from example 4. Recall that the price of the bond was $116.95. Now suppose that the price of the bond is $110.00 (that is, it is undervalued). Then, an investor could do the following:

- Sell the portfolio of zeros.

- Buy the coupon bond.

Period	**Cash Flow from selling**	**Cash Flow from buying**	**Total inflow**
0	+116.95	-110.00	116.95-110.00 = 6.95
1	-10	+10	0
2	-10	+10	0
3	-10	+10	0
4	-10	+10	0
5	-10	+10	0
6	-110	+110	0

Our investment resulted in a risk-free gain of $6.95 since we have made a sure profit at time 0. We have not entered into any future commitments, hence the $6.95 are a riskless arbitrage profit.

2.6 Bond Price Dynamics

Of considerable interest is how the present value of the cash flows from a bond investment varies with changes in the interest rate. Since we have expressed the present value in terms of the interest rate and the cash flows, the direction of change can be determined by the first derivative of the price function

$$B_0 = \left(1 + \frac{R}{m}\right)^{-N} \cdot F \qquad (2)$$

with respect to interest rates:

$$\frac{\partial B_0}{\partial R} = -\frac{N}{m} \cdot \left(1 + \frac{R}{m}\right)^{-N-1} \cdot F = -t \cdot \left(1 + \frac{R}{m}\right)^{-N-1} \cdot F \qquad (3)$$

Note that the sign of these derivatives is negative. This means that the price of the zero coupon bond or the present value of the cash flow will *decrease* with an *increase* in the interest rate. The derivative in (3) as an intuitive interpretation. It gives the change of the price (in currency) of the bond in response to a change in the level of the interest rate.[1] However, the derivative is only an approximation. Also, if we measure the change in the interest rate in percentage points, we need to divide (3) by 100.

Example 6

Consider a bond with a term to maturity of 5 years, a face value of $1,000 and assume that the interest rate is equal to 8%. Then we have

$$\frac{\partial B_0}{\partial R} = \$ - 1{,}000 * 5 * 1.08^{-6} * \frac{1}{100} = \$31.5085$$

Hence, if the interest rate changes by one percentage point, then the bond price changes by $31.51. Comparing this with the exact numbers, averaging across a 1% shift up to R=9% and a 1% down shift to R=7% gives:

[1] This is expressed by the ∂-symbol which denotes "difference" or "change": the derivative gives the change in the variable in the numerator in response to a change of the variable in the denominator.

Yield	Bond Value	$ Change	% Change
8%	680.5832		
9%	649.9314	-30.6518	-4.5038%
7%	712.9862	32.4030	4.7611%
Average		31.5274	4.6324%

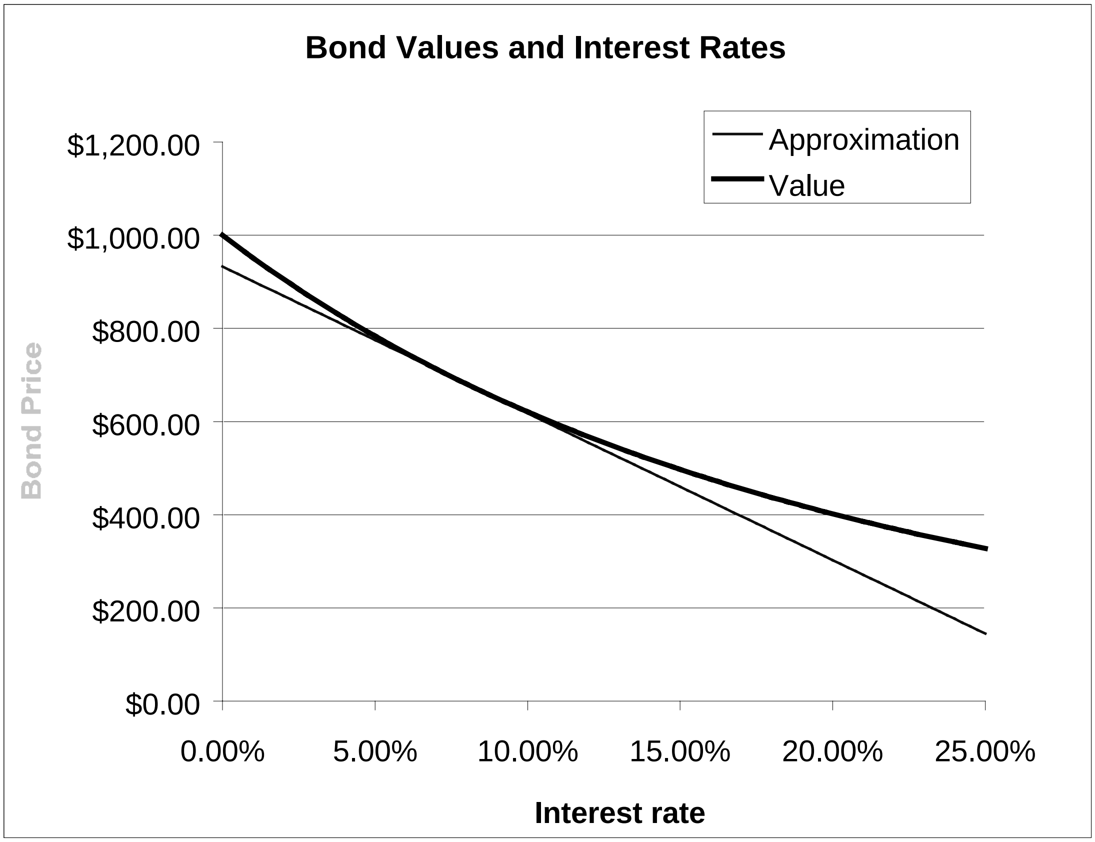
Hence, the errors for the case of an interest rate movement of one percentage point are somewhat small.

Notice that the time to maturity, *t*, affects the rate of decrease. A longer term zero coupon bond will decrease by more than a short term zero coupon bond. To gain additional insight, recall that the first derivative of a function is a first order (or linear) approximation to the slope of the function. Additionally, we can generally assume that such an approximation is accurate for *small* changes in the interest rate R. Hence, the price response ΔB_0 to a change in the interest rates by ΔR is approximately:

$$\Delta B_0 = -t(1+i)^{-N-1} F \cdot \Delta R \tag{4}$$

This expression gives us an approximation for the *absolute* price change in dollars ΔB_0 in

response to a shift in the interest rate by ΔR. We can demonstrate this graphically by looking at the plot of the bond price as a function of theinterest rate. The straight line gives us the approximation using a derivative. Hence, for small differences between the old interest rate (8% in the example) and the new interest rate after a shift in interest rates the approximation is good, whereas deviations are substantial if the interest rate moves by several percentage points.



We are often more interested in the *percentage* response of the bond price. We can obtain this by dividing the derivative by the value of the bond:

$$\frac{\partial B_0}{\partial R}\cdot\frac{1}{B_0}=\frac{-t(1+i)^{-N-1}\cdot F}{(1+i)^{-N}\cdot F}=-\frac{t}{1+i}$$
$$\Rightarrow \frac{\Delta B_0}{B_0}\approx -\frac{t}{1+i}\cdot\Delta R \qquad (5)$$

Here $\Delta B_0/B_0$ represents the *percentage* price change of the bond in response to a change by ΔR. Hence, this result says that the percentage price change of a zero coupon bond is proportional to the maturity of the bond.

Example 6 (cont.)

Reconsider the bond in example 6 above. The percentage change can be expressed as:

$$\frac{\partial B_0}{\partial R}\frac{1}{B_0} = 0.046296$$

or 4.63%. Hence, if the interest rate changes by 1%, then the bond price is expected to change by 4.63%. Compare this with the table above for the exact numbers and the magnitude of the mistake by taking derivatives.

2.7 Bond Valuation

Bond traders quote prices as a percent of par, with fractions in 32nds. For example, a price of 102-8 on a bond means 102+8/32=102.25% of par. If the par amount is $10 million, then the price is $10,225,000.

We will now introduce the general formula for pricing bonds. The price of a coupon-bearing bond can be written as follows:

$$B_0 = \sum_{n=1}^{N} \frac{c}{\left(1+\frac{R}{m}\right)^n} + \frac{F}{\left(1+\frac{R}{m}\right)^N} \qquad (6)$$

The price of the bond B_0 is simply the sum of the present values of all future payments.

Example 7

Reconsider the three-year coupon bond from example 4. We can use the general formula to price this bond assuming (as in example 4) that the semi-annual interest rate is 6.5%. It can be seen that the price of this bond is simply the sum of the present values of its coupons and face value. Recall that we have already calculated these present values when we replicated the cash flows of this bond using zero coupon bonds.

$$
\begin{aligned}
B_0 &= \sum_{n=1}^{6} \frac{10}{1.065^n} + \frac{100}{1.065^6} \\
&= \frac{10}{1.065} + \frac{10}{1.065^2} + \frac{10}{1.065^3} + \frac{10}{1.065^4} + \frac{10}{1.065^5} \frac{10}{1.065^6} + \frac{100}{1.065^6} \qquad (7) \\
&= 9.39 + 8.82 + 8.28 + 7.77 + 7.30 + 6.85 + 68.53 \\
&= 116.95
\end{aligned}
$$

Note that the general formula consists of two parts. The first is the annuity of N = 6 equal coupon payments. The second is the principal payment F = $100. Hence, we can rewrite the general valuation formula as:

$$B_0 = cA_n + 100B_n \qquad (8)$$

Where $\boldsymbol{cA_n}$ is the value of the n-period coupon annuity and $\boldsymbol{100B_n}$ is the present value of the principal payment of the n-period bond. In other words, we view the price of the bond as a sum of the present value of the coupon annuity and the present value of the final principal payment.

Example 8

Now we will calculate the price of a few bonds. Suppose that the current stated rate is 12.5% compounded semi-annually. There are two bonds in the market both mature in 12 years. Bond A has a 8.75% coupon rate (paid semi-annually) and Bond B has a 12.625% coupon rate (paid semi-annually). Before we start the calculations, it is clear that Bond B should be more valuable than Bond A. The coupon rate on Bond B is above the fair rate in the market and we expect it to be selling at a premium (above par). On the other hand, Bond A has a lower coupon rate and should be selling at a discount (below par). First, calculate the value of the one period zero.

$$i = \frac{R}{m} = 0.125/2 = 0.0625$$
$$B = \frac{1}{(1+i)} = \frac{1}{1.0625} = 0.941176 \quad (9)$$

The value of the 24 period zero B_{24} also needs to be calculated to bring the principal back to present value.

$$B_{24} = (0.941176)^{24} = 0.2334 \quad (10)$$

Now calculate the value of a one dollar annuity (*a* = \$1).

$$A_{24} = \frac{B \cdot (1 - B_{24})}{(1 - B)} = \frac{0.9412 \cdot (1 - 0.2334)}{(1 - 0.9412)} = 12.2665 \quad (11)$$

The coupons are easily calculated:

$c^A = 8.75/2 = 4.375$ and $c^B = 12.625/2 = 6.3125$

Now we can plug into our bond valuation formula:

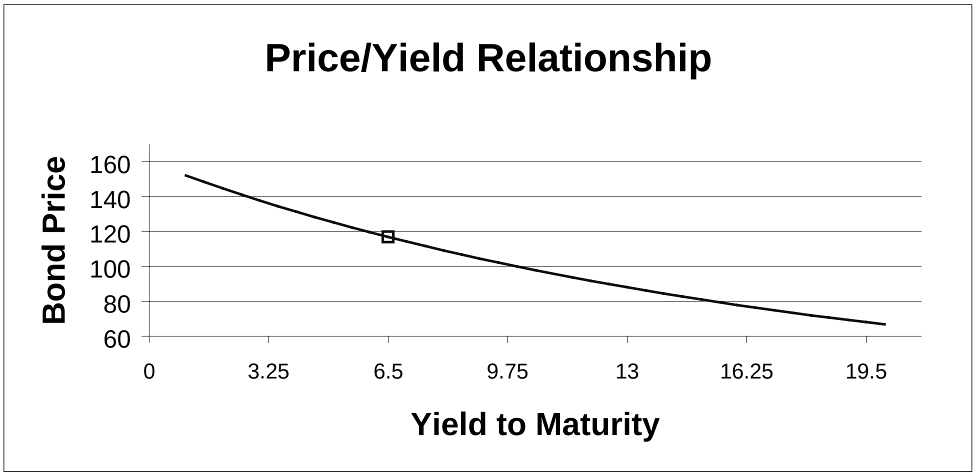
$B^A = (4.375)(12.2665) + 100(0.2334) = 53.66 + 23.34 = 77.00$

$B^B = (6.3125)(12.2665) + 100(0.2334) = 77.43 + 23.34 = 100.77$

2.8 Yield to Maturity

We now proceed to define a new concept, called the *yield to maturity*. Reconsider equation (6) above, where we discount future payments at the same interest rate R. When we price a bond, we take the interest rate as given, and determine the price by discounting. Now, we reverse the present value procedure. That is, given the bond market price, we *solve* for the interest rate that equates the present value of the cashflows to this price.

Look at the diagram below:



Previously, we took the interest rate or yield (on the horizontal axis) as given and calculated the price (on the vertical axis). The concept of a *yield to maturity* does the opposite: we take the price B_0 as given as a market price, and read the *yield* from the horizontal axis, i. e., we determine the yield as that particular discount rate that makes the present value of all future payments to the bondholder equal to the current market price.

Definition: The *yield to maturity* is the interest rate that *solves* the general pricing formula given the price of the bond:

$$B_0 = \left[\sum_{n=1}^{N} \frac{c_n}{1+\left(\frac{yield}{m}\right)^n} \right] + \frac{F}{\left(1+\frac{yield}{m}\right)^n} \tag{12}$$

Hence, given the price of the bond and its future cashflows, we solve for the interest rate that equates the price of the bond to the present value of these cashflows.

Note in particular that whenever the price of the bond is equal to its par value or principal value, then the yield must be equal to the coupon rate (compare equations (6) and (12) to understand why). If the current price of the bond is *higher* than the par value, we say the bond is trading at a premium. Then we must have that the market has used a discount rate lower than the coupon rate. Conversely, if the bond price is below its par value, we say it is trading at a discount, and the yield must exceed the coupon rate.

For a zero coupon bond we have

$$B_0 = F \cdot \left(1 + \frac{yield}{m}\right)^{-mt} \tag{13}$$

Solving for the yield we find:

$$yield = m \cdot \left[\left(\frac{F}{B_0}\right)^{\frac{1}{N}} - 1\right] \tag{14}$$

Similarly, with continuous compounding the bond price formula is

$$B_0 = e^{-yield \cdot t} \cdot F \tag{15}$$

Hence, the yield is (rearrange and take logarithms):

$$R = \frac{1}{t} \cdot \ln\left(\frac{F}{B_0}\right) \tag{16}$$

Example 9

Consider the zero coupon bond from example 1: The bond's a par value is $100, it matures in six years from now and is trading at $55. If interest were compounded annually, then the yield can be calculated as:

$$R = (100/55)^{(1/6)} - 1 = 10.48\%$$

If interest were compounded semiannually, then the yield can be calculated as:

$$R = 2((100/55)^{(1/12)} - 1) = 10.22\%$$

The continuous time yield is:

$$R=1/6\ln(100/55) = 9.96\%$$

Note that the more periods per year, the lower the yield. The yield with continuous compounding is always lowest. CATs, like treasuries and corporate bonds, are usually stated using two compounding periods per year. To avoid confusion, the yield is referred to as a semiannual yield.

There is no easy way to calculate the yield to maturity for a coupon-paying bond. Usually a computer will solve the equation numerically (using iterative methods).[2] There are advantages and disadvantages to using the yield to maturity. One advantage is that we are solving for the interest rate rather than plugging one in. It is also a widely used measure, (i.e., reported in the press).

Now we will illustrate some of the problems in using the yield to maturity. Suppose we have 2 bonds: Bond A and Bond B. Suppose they both cost $1000. Assume that they compound annually, rather than semiannually.

	Bond A	Bond B
Price (in market)	1000	1000
Cash Flows:		
Year 1	145	430
Year 2	145	430
Year 3	1145	430
Yield	14.5%	13.9%

Note that both of the bonds have the same scale costing $1000. Furthermore, they have the same investment horizon of 3 years. It appears as if Bond A is better - having a higher yield. But this is not necessarily the case.

In computing the yield to maturity we have assumed that the annual rates of return were equal. That is, the interest rate over period 1 was assumed equal to the interest rate over period 2 and 3.[3] But what if this was not the case? Suppose the interest rates prevailing over periods 1,2 and 3 were:

First year $= i_1 = 10\%$
Second year $= i_2 = 20\%$
Third year $= i_3 = 15\%$

That is, i2 = 20% means that the one year rate will be 20% in one year.

Now we calculate the present value using these rates:

$$B_A = \frac{145}{1.1} + \frac{145}{1.1 \cdot 1.2} + \frac{1145}{1.1 \cdot 1.2 \cdot 11.5} = \$996$$
$$B_B = \frac{430}{1.1} + \frac{430}{1.1 \cdot 1.2} + \frac{430}{1.1 \cdot 1.2 \cdot 1.15} = \$1000$$

From these calculations, the present value of Bond B is greater than the present value of bond A.

[2] **Excel** has an IRR function that solves the equation numerically. The *Solver* function in **Excel** can also be used.
[3] This pattern is also called a *flat term structure.*

Bond	YIELD	PV	Price
A	14.5%	996	1000
B	13.9%	1000	1000

It is clear from this example that Bond B is a superior investment to Bond A since its present value is higher. If we vary the pattern of i_1, i_2 and i_3, that is, vary the shape of the term structure, then the yield to maturity rule will not always work as a guide to higher returns.

2.9 Bond Rankings and Interest Rates

We have shown that the price of the bond is sensitive to the interest rate. Another factor that has to be taken into account when ranking bonds is the timing of the cash flows. If Bond B's cash flows are concentrated in the far future, then its price will be very sensitive to changes in interest rates. Conversely, if Bond A's cash flows are concentrated in the near future, it will not be as sensitive to changes in the interest rate.

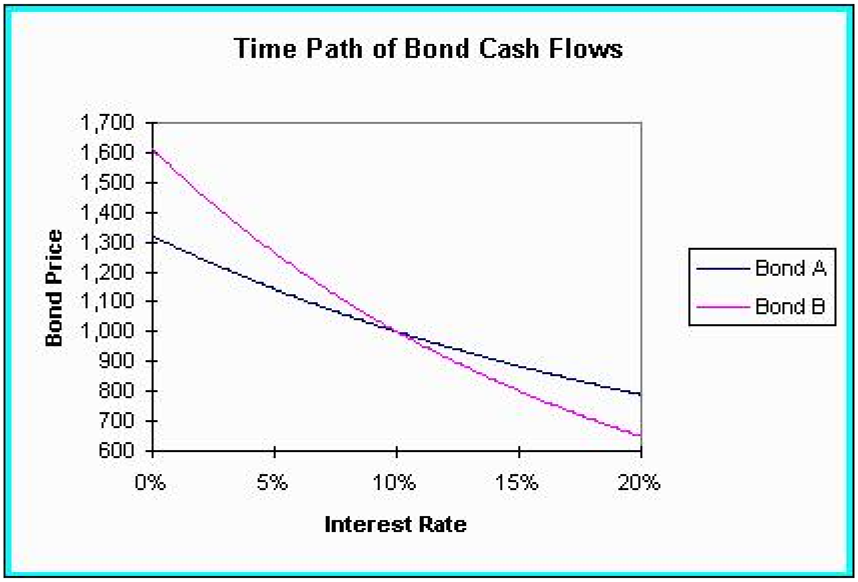
Consider the following example.

Year	Bond A Cash Flows	Bond B Cash Flows
1	263.80	0
2	263.80	0
3	263.80	0
4	263.80	0
5	263.80	1611.51

Now calculate the present values of these cash flows for various discount rates.

Yield	PV_A	PV_B	Better
0%	1320	1611	B
5%	1142	1262	B
10%	1000	1000	-
15%	884	801	A
20%	789	647	A

Hence, both bonds have the same value at a yield of 10%, A dominates B for higher, but not for lower yields. So the time path of cash flows is very important. Graphically, the present values



function for bond A intersects that for bond B from below at 10%. Hence, bond A (the coupon bond) is less sensitive to movements in interest rates than bond B (the zero coupon bond). In order to develop an intuition for this, remember our observation on zero coupon bonds: the sensitivity of bond prices to interest rates is proportional on the maturity of a *zero coupon bond.* It appears that for coupon bonds we have to take a slightly different approach, and observe that they are similar to a portfolio of zero coupon bonds, hence, the interest rate sensitivity of a coupon bond is a weighted average of the interest rate sensitivity of all these zero coupon bonds. However, the maturity of the zero coupon bonds in the replicating portfolio that match the coupon payments is *less* than the maturity of the coupon bonds. Hence, we have the general and very important result:

The interest rate sensitivity of a coupon bond is also always *less* than that of a zero coupon bond with the same maturity.

In fact the interest rate sensitivity of a coupon bond is proportional to the *average maturity* of the zero coupon bonds in the replicating portfolio. This "average maturity" is called *duration* and plays an important part in the hedging of bond portfolios.[4]

2.10 Real Interest Rates and Nominal Interest Rates

So far we talked about interest rates in terms of currency, i. e., US dollars, pound sterling, and so forth. However, investors are ultimately not interested in receiving dollars or pounds, but in the rate at which they can increase their consumption in the future if they forego some consumption today. This is reduced through the impact of inflation. Obviously, the impact of inflation depends on the prices of the individual goods each investor wishes to purchase. We take a standard approach here and measure inflation through changes in the consumer price index (CPI), which implicitly assumes that all investors are interested in buying consumption goods in proportion to the weights of the index. This procedure has a number of limitations that are not the proper subject of this course.

The CPI is conventionally expressed as an index value that is equal to 100 for some base year. Consider an investor who contemplates consuming a certain amount of money either today, or exactly one year from now. Assume the risk free rate of interest is 7.5%, and the value of the CPI today is 125. Suppose that we can forecast inflation with certainty (unfortunately, this is never the case, but it spares us some complications here). The expected value of the CPI at the end of the year is known to be 130. Then the investor can *purchase* the bundle of commodities representing the index for $125 today. Alternatively, she can save the $125, and invest them at the current rate of interest to obtain $125*1.075=$134.38 in one years time, and purchase then 134.38/130=1.0337 units of the index. Hence, if the investor defers consumption by one year, she can increase the amount she consumes by a factor of 1.0337, i. e., she can consume 3.37% more than if she consumes today. This percentage is called the *real rate of interest*, since it properly reflects the *real return* of the investor, i. e., how much she can consume more by deferring consumption for one year. The difference between the nominal and the real return is simply inflation. Here the inflation rate is assumed to be 130/125-1=4%. We use the following notation for the general case:

rr_t	Real rate of interest at time t
CPI_t	Value of the consumer price index at time t
R	nominal interest rate
$\pi_t = \frac{CPI_t - CPI_{t-1}}{CPI_{t-1}}$	Inflation rate at time t

For the general case, if the investor considers consuming *$X* today, then she can either consume X/CPI_0 units of the consumption basked today, or $(1+r)X/CPI_1$ units of the consumption basket in the future, where r is the nominal interest rate between these two points in time. Hence, the real rate of interest is:

$$1 + rr = \frac{(1+R)X / CPI_1}{X / CPI_0} = (1+r)\frac{CPI_0}{CPI_1} = \frac{1+R}{1+\pi} \tag{17}$$

For small interest and inflation rates we can rewrite this formula as:

[4] We do not develop duration in the context of this course. Chapter 22 of Grinblatt and Titman, Financial Markets and Corporate Strategy, McGraw-Hill (1997) is a readable introduction.

$$rr \approx R - \pi \tag{18}$$

In our example, we can compute the inflation rate as:

$$\pi = \frac{130 - 125}{125} = 4\% \tag{19}$$

which gives us a real interest rate of $\frac{1.075}{1.04} = 1.0336$, whereas the approximation would give us 7.5%-4%=3.5%.

Some governments have started to issue index-linked bonds. These bonds have coupons that are linked to the price index: if inflation is 5% in any particular year, then the coupon and principal payments of such a bond are increased by 5%, hence investors are given a protection against inflation.

2.11 Forward Interest Rates

A *forward interest rate* is the rate of return for investing your money for an extra period, i.e., investing for *t* periods rather than *t-1* periods. For simplicity let a "period" be one year and r_0^1 and r_0^2 be the annualized interest rates prevailing between today and next year and between

periods today and year two. Then the annualized forward rate between periods year one and two f_1^2 satisfies the following relation:

$$\left(1 + r_0^2\right)^2 = \left(1 + r_0^1\right) \cdot \left(1 + f_1^2\right) \tag{20}$$

This rate answers the following question: if you invest \$1 for one year, then your return is simply $\$1+r_0^1$. If you invest \$1 for two years, your return is $\$\left(1+r_0^2\right)^2$. We are interested in *how much more* you receive by investing for one *more* year, and this is the forward rate f_1^2: it is the rate of investing \$1 between year 1 and year 2. Solving for the forward rate is quite easy (rearrange (20)):

$$f_1^2 = \frac{\left(1+r_0^2\right)^2}{\left(1+r_0^1\right)} - 1 \tag{21}$$

We can also calculate multiperiod (annualized) forward rates. Let r_0^3 and r_0^6 be the annualized interest rates prevailing between today and year three and between today and year six. Then the annualized forward rate between years three and six f_3^6 satisfies the following relation

$$\left(1+r_0^6\right)^6 = \left(1+r_0^3\right)^3 \cdot \left(1+f_3^6\right)^3 \tag{22}$$

We demonstrate the interpreation of forward rates in the next example.

Example 11

Suppose we look in the paper and find a one year zero (face value \$100 million) trading at \$92.59 million (yield of 8% annual rate no compounding) and a two year zero trading at \$79.72 million (yield of 12% annual rate no compounding). Consider the following strategy.

- We *sell* (issue) \$100 million face value of the one year bond and in turn we pocket today the price of the bond \$92.59 million.

- We use the proceeds ($92.59) to purchase as much of the two-year bond as possible. We are able to purchase {$92,592,590/$79.71938}= 1.161480 of these bonds.

- At the end of the first year, we pay the purchaser of the one year bond $100 million.

- At the end of the second year, we realize the revenue from cashing in the two-year bonds. That is we redeem the bonds for $116,148,000.

- Effectively, we have a zero cash flow today, a cash outflow of $100 million in year 1, and a cash inflow of $116.148 million in year 2. Hence, we have invested between year 1 and year 2. (see table below).

- The one year return from years one to two is (116.148-100)/100=16.148%. This is exactly the definition of the forward rate from year one to year two. To verify this recall from equation (21) that the forward rate f_1^2 satisfies the following equation:

$$f_1^2 = \frac{\left(1+r_0^2\right)^2}{\left(1+r_0^1\right)} - 1 \tag{23}$$

Given the yields on the one and two year bonds we can solve this equation

$$f_1^2 = \frac{\left(1+.12\right)^2}{\left(1+.08\right)} - 1 = .16148 \tag{24}$$

Hence, *the forward rate is also the return to an investment strategy that involves selling and buying bonds of different maturities*.

The following table describes these transactions:

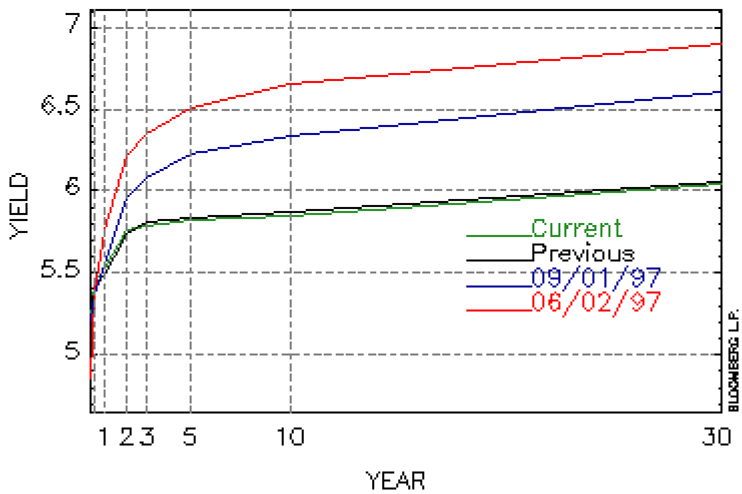
Action	Inflows Today	Period 1	Period 2
Short one year bond	+$92.59m	-$100m	0
Buy two year bond	-$92.59m	0	+$116.148m
Total	0	-$100m	+$116.148m

2.12 The Term Structure of Interest Rates

The term structure of interest rates or the yield curve is the relation between yields observed today on bonds of different maturity. Consider the following table:

Year	1	2	…	T
Yield	R_1	R_2	…	R_T
Example a	7%	8%		10%
Example b	8%	8%	8%	8%
Example c	9%	8%		7%

Graphically, the yield curve is the curve obtained by plotting R_1, R_2,…,R_T. The yield curve is upward sloping if longer term bonds have higher yields than shorter term bonds or Treasury bills, as in example a. The curve is flat if all the yields are the same (example b). The structure is inverted if yields on short-term bills are higher than long term bonds (example c). The term structure for the US is given in the picture (obtained from the web site of bloomberg, (http://www.bloomberg.com/markets) click on "Treasury yield curve") You can see that interst rates have fallen, but they have fallen more at the "long end" of the curve (longer maturities) than at the "short end".



There have been many theories proposed to explain the term structure of interest rates. The three main theories that you probably studied in your macro course are: expectations, liquidity preference and preferred habitat. The expectations theory just says that a positively sloped yield curve means that investors expect rates to go up. Liquidity preference suggests that a rate premium be attached to longer term bonds because they are more volatile. The preferred habitat says that different rates across different maturities are due to differential demand by investors for particular maturities.

Acknowledgement

Much of the materials for this lecture are from Douglas Breeden, *"Interest Rate Mathematics"*, Robert Whaley, *"Derivation and Use of Interest Formulas"* and Campbell R. Harvey and Guofu Zhou, *"The Time Value of Money"*.

Important Terminology

Important Formulae

Price of a zero-coupon bond:

$$B_0 = \frac{F}{(1 + R/m)^N} \tag{1}$$

Coupon-bond:

$$B_0 = \sum_{n=1}^{N} \frac{c}{\left(1 + \frac{R}{m}\right)^n} + \frac{F}{\left(1 + \frac{R}{m}\right)^N} \tag{6}$$

Yield to maturity of a zero coupon bond:

$$yield = m \cdot \left[\left(\frac{F}{B_0} \right)^{\frac{1}{N}} - 1 \right] \tag{14}$$