Global Financial Management

Discounting and Present Value Techniques

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1.0 Overview

In this lecture we will introduce discounting techniques and interest rate mathematics. This material is important in several respects. The techniques are fundamental for almost any financial calculation, ranging from simple tasks like calculating the repayments on a mortgage or tracking a loan balance to more complex applications. The material is also foundational for subsequent lectures, particularly bond valuation, stock valuation and investment appraisal techniques. Finally, the discussion below offers some basic insights into the main tasks and functioning of financial markets, which we are going to deepen during the remainder of the course.

1.1 Objectives

At the end of this unit you should be able to:

- Track a loan balance.
- Decide whether you should re-mortgage your house.
- Determine the required monthly contribution to your pension plan.
- Decide what lump sum you need to set aside today in order to fund the college education of your children.

- Value a perpetual bond.

- Distinguish annual percentage rates and effective annual rates and use them correctly.

1.2 Future Value: The value tomorrow of a dollar today

Suppose you just received a bonus payment of $10,000, and you can put it into a bank account at a rate of 6% p. a. (p. a. = per year). Planning ahead, you want to determine the consumption you can afford one year from now. How does this compare to receiving a bonus payment of $10,000 one year from now? One of the most fundamental principles of finance is that a dollar today (or $10,000, in this case) is worth more today than a dollar tomorrow. Let’s look at these numbers in more detail. You can invest the $10,000 at 6%, so at the end of one year the sum in your savings account has grown to $10,600: you own $600 more a year from now if you receive the $10,000 today rather than a year hence. How would this relationship change, if you compare a $10,000 today with a payment of $10,000 two years from now? We know already that we obtain $10,600 at the end of the first year if we invest them in the savings account for one year. At the end of the second year we will have:

$$\$10{,}600 * 1.06 = \$11{,}236 \ .$$

We call the number $11,236 the **future value** of $10,000 at 6%, 2 years from now. We wish to develop the concept of a future value more generally. Observe that the future value can be obtained as follows:

$$\textit{future value} = \$10{,}000 * 1.06 * 1.06 = \$10{,}000 * 1.06^2 \ .$$

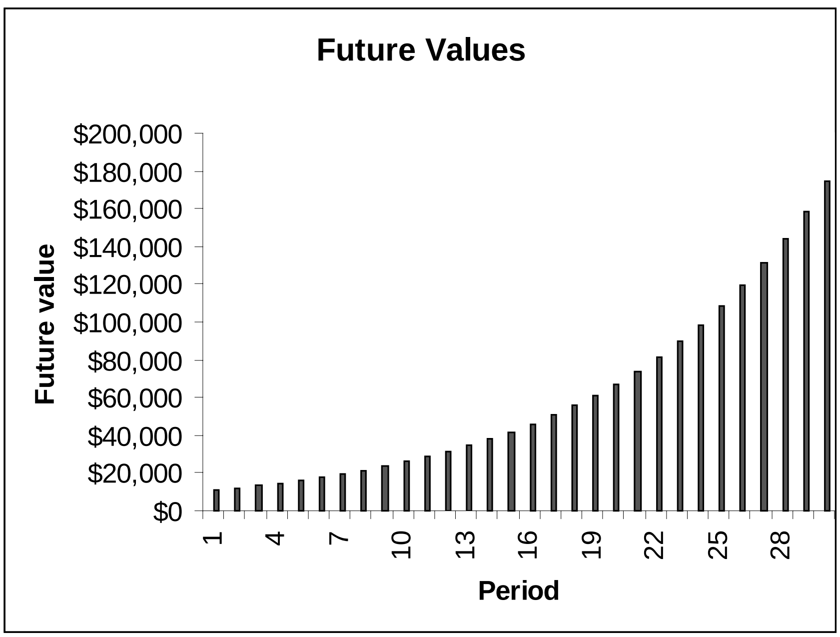
We can iterate this process in order to find out what the future value of $10,000 today at 6% is at the end of three, five or ten years, and how it would change if the interest rate were 8%. We use the following symbols:

V_n	Future value after n years
V_0	Initial investment
i	Interest rate per year

Then we can write down a general expression for future value:

$$V_n = V_0 (1+i)^n \ . \tag{1}$$

Suppose V_0=$10,000 and i=10%. Then we obtain the following pattern of future values for n ranging from 1 to 30.



After 30 years, our initial value of $10,000 has grown to a respectable

$\$10,000*1.1^{30}=\$174,494$.[1]

Consider also the following example.

Example 1:

Suppose you obtain two payments, $5,000 today and $5,000 exactly one year from now. You can put these payments into a savings account and earn interest at a rate of 5%. What is the balance in your savings account exactly 5 years from now? Compute the following table:

Year	Cash inflow	Interest	Balance
0	$5,000.00	$0.00	$5,000.00
1	$5,000.00	$250.00	$10,250.00
2	$0.00	$512.50	$10,762.50
3	$0.00	$538.13	$11,300.63
4	$0.00	$565.03	$11,865.66
5	$0.00	$593.28	$12,458.94

At the end of the first year you receive $250 interest (5% of $5,000), giving you a total balance of $10,250 one year from now. Then you can compute future values for each year as $\$10,250*1.05^n$, where n=1 for the balance two years from now, and n=4 for the balance 5 years from now, so in five years you have a balance of:

$\$10,250*1.05^4=\$12,458.94$.

You can easily extend the above example to other applications in order to keep track of the future development of an investment that you make today.

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[1] This and the following calculations were done in a spreadsheet program. Please refer to the spreadsheet. Replication of the numbers in the examples with a calculator may lead to slightly different results because of rounding errors.

1.3. Present Value: The value today of a dollar tomorrow

Often you will need to do the reverse operation of the one we performed above, and ask the question: How much do you have to put into your bank account today, so that in one year from now, the balance is exactly \$10,000, if I accrue interest at a rate of 6% on the balance. Hence, you wish to determine the amount *P* that solves:

$$P * 1.06 = \$10{,}000$$

which gives you immediately:

$$P = \frac{\$10{,}000}{1.06} = \$9{,}433.96$$

Hence, if you put in \$9,433.96 into your bank account today, then this amount will grow into exactly \$10,000 one year from now. We call the amount *P* (here \$9,433.96) the **present value** of \$10,000 one year from now at an interest rate of 6%. It is easy to see from equation (1) above that if V_n is the future value (i. e., the end of period balance in our savings account), then V_0 is the present value. We can therefore solve (1) for the present value V_0 to find:

$$V_0 = \frac{V_n}{(1+i)^n} \qquad (2)$$

The present value formula (2) presents the flip side of the principle that a dollar tomorrow is worth less than a dollar today. In our case, one dollar in a year from now is worth only \$0.9433. Present values are important if you wish to compare different liabilities. Consider the following example:

Example 2:

Suppose your daughter Jane just graduated from college and wishes to take a postgraduate course. Jane has the choice between two universities of comparable quality that offer the two-year course of her choice. University A charges \$8,000 of tuition fees for the first year and is expected to increase these fees to \$10,000 in the second year, whereas university B charges \$9,000 in the first year and \$9,500 in the second year. You give Jane a sum sufficient to cover all her tuition fees, and she can invest this at a rate of 5.5% p. a. How much do you have to give to Jane if she attends the course at either university A or B? Assume all tuition fees are always due at the end of the year.

We use a step-by-step approach to calculate the amount Jane has to borrow if she attends the course at A. The present value of a payment of \$8,000 at 5.5% at the end of one year is:

$\$8{,}000/1.055=\$7{,}582.94$.

The present value of a payment of \$10,000 at the end of the second year is:

$\$10{,}000/1.055^2=\$8984.52$.

Hence, if Jane chooses A, you have to give her \$16,567.46 today to fully cover her tuition fees. Similarly, for B we find:

$$\frac{\$9{,}000}{1.055}+\frac{\$9{,}500}{1.055^2}=\$17{,}066.10 \ .$$

Let’s check that this actually works. If Jane chooses B and you hand over the sum of \$17,066.10, she puts it into her savings account and accumulates a balance of

$18,004.74=$17,066.10*1.055

at the end of the year. Then she pays $9,000 to the university and retains $9,004.74 in her account. At the end of the second year she has accumulated

$9,004.74*1.055=$9,500.00

in her savings account, just enough to meet the bill for tuition fees at the end of the second year. The following table tracks thepayments and balances of jane's account.

Date	Initial Balance	Payment	Remaining Balance
0	$17,066.10	$0.00	$17,066.10
1	$18,004.74	$9,000.00	$9,004.74
2	$9,500.00	$9,500.00	$0.00

The last example introduces another aspect of present values that is worth emphasizing. We could simply add the present values of the two payments for tuition fees. This property is called **value additivity** and makes working with present values very straightforward. To generalize this intuition, we use the symbol C_t for the cash flow at the end of year t. In the previous example, we have C_1=$9,000 and C_2=$9,500 for university B. Then we define present value of a stream of payments $C_0 \ldots C_n$ as:

$$V_0 = C_0 + \frac{C_1}{(1+i)} + \frac{C_2}{(1+i)^2} + ... + \frac{C_{n-1}}{(1+i)^{n-1}} + \frac{C_n}{(1+i)^n} = \sum_{t=1}^{t=n} \frac{C_t}{(1+i)^t} \qquad (3)$$

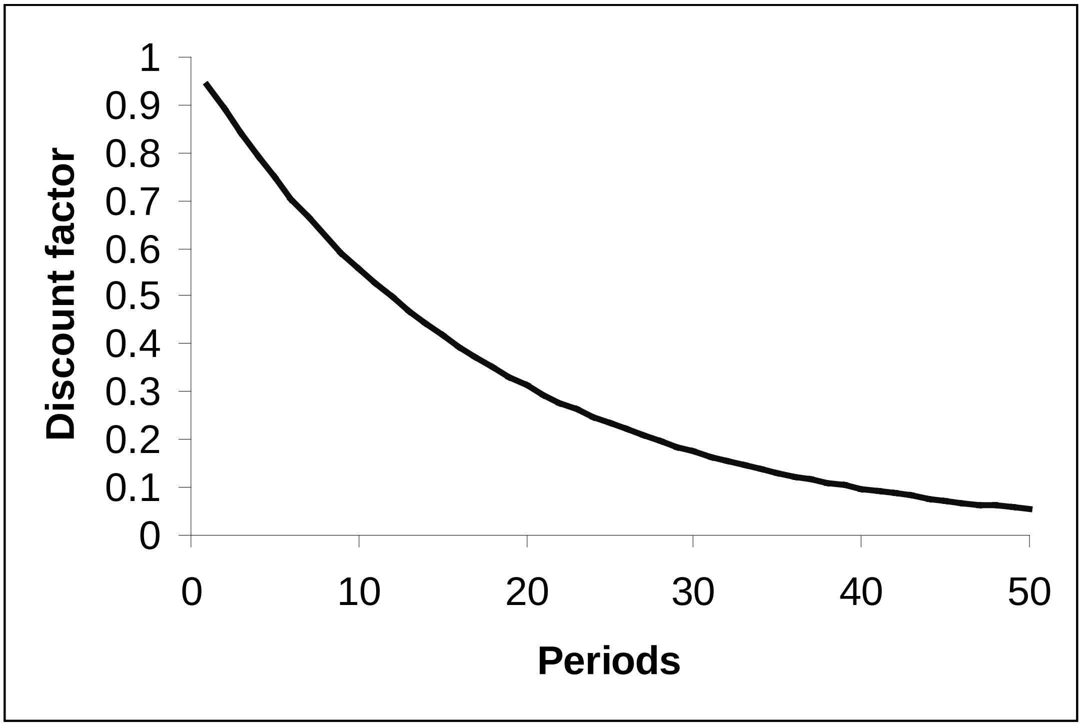
which shows that we can always calculate the present value of a series of payments by adding the present values of each individual payment.[2] Note that each element of the

2 The Greek letter Σ denotes the so-called summation operator and reads as follows: Sum a series of elements displayed on the right hand side of the letter Σ for all t

series in (3) has a very simple structure. We multiply the payment C_t in period t by the factor $\frac{1}{(1+i)^t}$ which depends only on the date of the payment and the interest rate. This factor is called the **discount factor**, and if we need to be more precise, we refer it also as the t-period discount factor. The following table and graph give the discount factor for an interest rate of 6% for up to five periods:

Period $=t$	**Discount factor** $=1/1.06^t$
1	0.943
2	0.890
…	
23	0.262
24	0.247
25	0.233
…	
49	0.058
50	0.054

between 1 and n. Hence, the right hand side of equation (3) is a shorthand for the left hand side of the equation.



Example 3:

Reconsider the tuition fee example, but assume that university A requires that half of the tuition fee for a particular year is paid before and the other half at the end of the academic year. Hence, Jane has to make the following payments:

	Year	Payment
C_0	0	\$4,000
C_1	1	\$9,000=\$4,000+\$5,000
C_2	2	\$5,000

Note that the payment at the end of the first year covers half of the tuition for the first and half of the tuition for the second year. Hence, now the amount of money she has to put aside to cover her tuition at university A is:

$$\$4{,}000 + \frac{\$9{,}000}{1.055} + \frac{\$5{,}000}{1.055^2} = \$17{,}023.07$$

almost as much as the present value of tuition fees for B.

The main advantage of present values is that they make payment streams with different timings comparable. In our example, the payments at university A are due at times 1 and 2, those for B at 0, 1 and 2. In order to compare like with like, we need to express these payments in one common unit, here today's dollars. However, the principle is only to express payments in terms of dollars of the same year, we could equally well choose the last year. Suppose we apply formula (1) to the present value expression in equation (3). We obtain:

$$V_n = \sum_{t=0}^{t=n} C_t (1+i)^{n-t} \; . \qquad (4)$$

We derive the steps leading to (4) in the appendix. Expression (4) has an intuitive interpretation as a loan balance.[3] Suppose you need to make a sequence of payments $C_0 \ldots C_2$ in order to meet an obligation (like college tuition fees), and you have to borrow the money you need for these payments from the bank because you have no income. Then the value V_n is the balance on your loan on the day after you made the last payment C_n. To see this, note that you compound interest on the first payment over n years, so after n years you owe $C_0(1+i)^n$ for borrowing C_0. On payment C_1 you compound interest only over n-1 years, so you owe $C_1(1+i)^{n-1}$ at the end of year n. Continuing this way, after the n-th payment of C_n (on which you have not accrued any interest), your loan balance is given by (4).

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3 Effectively, there is no difference whether you analyze this from the point of view of a borrower as a loan balance, or from the point of view of a lender as a fund into which you make period payments.

Example 4:

For our example we can express the payment streams to A and B in terms of their end of year 2 values by using the future value formula:

$$\$17{,}023.07 * 1.055^2 = \$18{,}947.10$$

Suppose you did not give any money to Jane, and she had to take out a loan to cover her tuition. Then she would make the required payments as described in example 3. On the first payment of $4,000 at the beginning of the first year she owes $4,000*1.055=$4,220 at the end of the first year. Then she makes another payment of $9,000, adding this amount to her loan balance Over her second year she has to pay interest on the $4,220, so her liability from this payment increases to

$\$4{,}220 * 1.055 = \$4{,}452.10 = \$4{,}000 * 1.055^2$.

Add to this the principal and interest from the second payment, which amounts to $9,000*1.055=$9,495.00 to give a total of $13,947.10 Adding the last payment after year 2 of $5,000 gives a total loan balance of $18,947.10, which is exactly the future value we just computed. The following table gives the balances and payments:

Year	Initial Balance	Payment	Interest	Balance
0	$0.00	$4,000.00	$0.00	$4,000.00
1	$4,000.00	$9,000.00	$220.00	$13,220.00
2	$13,220.00	$5,000.00	$727.10	$18,947.10

Hence, we can summarize that present values and future values are useful in order to compare payment streams. The important principle is value additivity: we can add present values and future values, provided they are expressed in terms of dollars of the same year. Present values have the interpretation of money to be set aside (for a liablity), or wealth in terms of current dollars (for an asset). If I have to make payments $C_0 \ldots C_n$

over the next n years, and I can accumulate interest at a rate i % per year, then the lump sum I have to set aside today in order to meet these obligations is the present value of these payments V_0. Similarly, future values keep track of account or loan balances.

If I have to make payments $C_0 \ldots C_n$ over the next n years, and I can borrow the money and accumulate interest at a rate i % per year, then the loan balance after making the last payment is the future value from these payments V_n. Similarly, if I make payments into a savings account and accumulate interest, then the balance of my savings account is given by the future value of all payments into this account.

1.4. Compounding Intervals

So far we have made one limiting assumption by maintaining that interest is compounded annually. This is unlikely, and different financial contracts come with different compounding intervals: mortgage and credit card loans typically compound interest monthly, savings accounts quarterly, and bonds semi-annually. Hence, we modify our symbols as follows:

t	Number of years
m	Number of compounding periods per year
$n=m*t$	Number of compounding periods
R	Nominal or stated interest rate, also called the APR (annual percentage rate)
$i=R/m$	periodic interest rate
r	effective annual interest rate

Example 5:

Suppose you have a 25-year mortgage with a stated APR of 9%, where interest is compounded monthly. Then t=25, m=12 and the number of compounding periods is n=12*25=300 months. The APR is R=9%, and i=0.75% is the monthly interest rate.

Example 6:

You take out a loan to finance your car at an interest rate of 12%, with 20 quarterly payments. Then n=20, m=4 and t=20/4=5 years. The APR is R=12%, i=3% is the quarterly interest rate.

In order to introduce the concept of an effective annual rate, we need to study the impact of different compounding intervals a little more closely. The most straightforward case here is future value. Suppose you take out a loan of V_0 dollars today with an APR of R and n compounding intervals. Then formula (1) above is still valid, but n is now the number of compounding intervals, which is generally not the number of years, and i is the interest rate per compounding interval, not per year. In order to express this in terms of years and annual percentage rates, we can substitute to obtain:

$$V_n = V_0 * (1+i)^n = V_0 * \left(1+\frac{R}{m}\right)^{m*t} \tag{5}$$

Example 7:

Suppose you borrow $10,000 at an APR of 12% and repay it in one lump sum at the end of the year. If interest is compounded annually, then you owe $11,200 at the end of the year. However, if interest is compounded semi-annually, then your interest rate for half a year is 6%, so your loan balance after six months is $10,600. Therefore, at the end of the year you need to repay

$10,600*1.06=$11,236.00.

The additional $36.00 represents the **compound interest** (6% of the $600 interest added to your loan balance after 6 months). Similarly, with quarterly compounding your loan balance would accumulate to $11,255.09. The following table gives your liability for different compounding intervals if you repay after one or after two years in one lump sum:

Compounding Period	**Compounding Intervals**	**1**	**2**
Year	1	$11,200.00	$12,544.00
6 months	2	$11,236.00	$12,624.77
4 months	3	$11,248.64	$12,653.19
Quarter	4	$11,255.09	$12,667.70
2 months	6	$11,261.62	$12,682.42
Months	12	$11,268.25	$12,697.35
Days	365	$11,274.75	$12,711.99
Hours	8760	$11,274.96	$12,712.47
Seconds	525600	$11,274.97	$12,712.49

We can see immediately that increasing the number of compounding periods also increases the effective costs of a loan.

Hence, if the compounding interval is not one year, then the APR does not give us the interest costs of the loan for one year any more. This leads us to the definition of an **effective interest rate** which we denote by *r*, and which is different from the **stated or nominal interest rate** or **APR**. To see the difference, consider the case of a monthly compounding period in example 7 and the previous table. The stated interest rate (APR) is 12%. The interest that accumulates on the loan is the same as if we had annual compounding and an interest rate of 12.6825%, substantially higher than the stated rate of 12%. (see the shaded row in the table) This interest rate of 12.6825% is our effective annual rate. It is defined as the rate that we need to apply to the original loan in order to obtain the total interest that accumulates on the loan in one year. Note that this rate also works for two or any number of years. In our example, after two years we have accumulated \$10,000*1.126825^2 =\$12,697.35. The effective annual rate is what we need in order to compute the effective interest costs of the loan. Hence, it must satisfy:

$$V_n = V_0 * (1+r)^t = V_0 * \left(1+\frac{R}{m}\right)^{m*t} \tag{6}$$

which gives us immediately that:

$$r = \left(1+\frac{R}{m}\right)^m - 1 \tag{7}$$

which is derived in the appendix.

Example 8:

For the numbers in example 7 we obtain for the effective annual rate:

Compounding Period	Compounding Intervals	Effective annual interest rate
Year	1	12.0000%
6 months	2	12.3600%
4 months	3	12.4864%
Quarter	4	12.5509%
2 months	6	12.6162%
Months	12	12.6825%
Days	365	12.7475%
Hours	8760	12.7496%
Seconds	525600	12.7497%

One observation is immediate from this table: the effective interest rate increases with the number of compounding intervals, but it does so at an ever smaller rate. As the number of compounding intervals becomes infinitely large (or, equivalently, as the length of one compounding interval becomes infinitesimally small), we can find a very convenient expression for the effective annual rate as:

$$r = e^{R} - 1 \tag{8}$$

where *e* represents Euler's number (*e*=2.71828).[4] Note that you have to express the interest rate in decimal form here, e. g., 12% as 0.12. In our example we obtain:

$$r = e^{0.12} - 1 = 0.127497$$

This is the same number we obtain with compounding every second up to 6 decimal places. In markets where we have to work with daily intervals (e. g. futures and options markets) continuous compounding is most of the time easier than computing effective annual rates from (8).

Example 9:

You have to make a payment on a loan with a current balance of $100,000 that matures 115 days from now. Interest accumulates daily on this loan at a rate of 6% p. a. What is the effective annual rate on this loan? What error do you make in your calculation if you assume that interest is compounded continuously? If interest accumulates daily, then the effective annual rate is:

$$r = \left(1 + \frac{0.06}{365}\right)^{365} - 1 = 0.061831$$

or 6.1831%. This gives a loan balance in 115 days from now of

$$\$100{,}000 * \left(1 + \frac{0.06}{365}\right)^{115} = \$100{,}000 * (1 + r)^{\frac{115}{365}} = \$101{,}908.23$$

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4 Note that *e* is also the base of the natural logarithm, commonly denoted by *ln*: *ln*(e)=1.

With continuous compounding we obtain:

$$r = e^{0.06} - 1 = 0.061837$$

$$\$100{,}000 * e^{0.06*\frac{115}{365}} = \$101{,}908.39$$

a difference of 16 cents on a loan balance of $100,000.

Example 10:

Suppose the effective annual interest rate is 9%. Which APR do you have to use if interest on this account accumulates monthly? Continuously? This is a more advanced question. We need an unknown interest rate *R* such that:

$$\left(1 + \frac{R}{12}\right)^{12} = 1.09 \Rightarrow R = \left(\sqrt[12]{1.09} - 1\right) * 12 = 0.0865$$

or 8.65%. The operation for continuous compounding involves taking logs:

$$e^{R} = 1.09 \quad \Rightarrow \quad R = \ln(1.09) = 0.0862$$

or 8.62%.

Note that you have to be careful with expressing the interest rate and calendar time here. *R* is an annual rate, hence time has to be expressed in fractions of one year.[5]

Example 11:

Suppose you accrue interest on your credit card continuously at a rate of 1.5% per month. What is the effective monthly rate? What is the effective annual rate?

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5 If we had expressed the interest rate for another compounding interval, e. g., as a monthly rate, then we would have to express time in terms of months. This is simply a requirement to kep the units of measurement consistent.

What would it be if interest compounded monthly? Hence, how much interest do you accrue on a balance of $1,000 if you repay after 6 weeks (42 days)? Compute the effective monthly rate first as $e^{0.015} - 1 = 0.01511$. Compounding over 12 months gives

$1.01511^{12}-1=1.1972-1=0.1972$

or 19.72%. If interest compounded monthly the effective annual rate would be only $1.015^{12}-1=0.1956$, or 19.56%. To compute your loan balance after 42 days with continuous compounding, use:

$$1.1972^{\frac{42}{365}} * \$1{,}000 = \$1{,}020.93 \ .$$

1.5. Discounting with an infinite time horizon

Equations (3) and (4) sum up all the concepts regarding present values and future values. However, in some cases it is possible to simplify these expressions if the stream of payments $C_0 \ldots C_n$ has a certain pattern. Surprisingly, the easiest formula obtains in the case where payments (i) are constant, and (ii) continue indefinitely into the future. This pattern is known from so-called Consols. These are perpetual bonds ("consolidation bonds") issued by the British government in the 19th century that have a constant coupon and are never repaid.[6] Another application is company valuation, where we cannot assume that dividend payments stop at a definite point in time.[7]

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[6] We will have more to say about these in the lecture on bond valuation.

[7] We will discuss this in more detail in the lecture on stock valuation.

Therefore, assume that $C_t=C$ for all periods t starting at 1 (i. e., starting at the end of the current period) for all periods into the future, and also assume that i is the appropriate discount rate. Then it turns out that the present value of this stream of perpetual payments is:

$$V_0 = \sum_{t=1}^{t=\infty} \frac{C}{(1+i)^t} = C * \sum_{t=1}^{t=\infty} \frac{1}{(1+i)^t} = \frac{C}{i} \ . \tag{9}$$

We derive this expression in the appendix using a standard formula for the summation of geometric series.

Example 12:

Suppose you are offered a perpetual bond that gives you one annual payment of \$50 at the end of each year, and the next payment is exactly one year from today. The appropriate discount rate is 4% p. a. How much are you willing to pay for this bond? Applying equation (9) gives:

$$V_0 = \frac{\$50}{0.04} = \$1{,}250 \ .$$

You may find equation (9) more intuitive by expressing it as:

$$C = i * V_0 \ .$$

This can be given the following interpretation. Suppose you borrow V_0, and you pay the interest due on this loan at the end of each period, but you never make a repayment of principal. Then C is your interest payment at the end of each year. Of course, if you never repay any principal, then you have to keep making interest payments C at the end of each year indefinitely.

Example 13:

Suppose you take out a loan of $2,000 at a rate of 7.5% with annual compounding, and do not repay any principal and make annual interest payments at the end of the year. Then you pay the lender $150=0.075*$2,000 at the end of each year, and you always owe the principal.

1.6 Annuities and Mortgages

Constant perpetual payments are easy to analyze, but they are not very common. A more common payment pattern is a so-called **annuity**, where payments are also constant, but extend over a finite period of time. The most frequent example for this is a mortgage loan, where the borrower repays the lender a loan in a specified number of equal instalments.

Example 14:

You take out a loan of $150,000 on your house at a mortgage rate (APR) of 6% over 30 years. This means that you repay the bank by making 360 monthly payments *C*. What is the monthly repayment *C* on this loan? Note that i=0.06/12=0.5%, so, clearly, the answer is more than $750 (=0.5% of $150,000), since $750 would only repay the interest, but not repay any principal. We need to determine how much more.

Our objective is now to value a stream of n constant payments *C*. Applying (3) once more gives:

$$V_0 = C * \sum_{t=1}^{t=n} \left(\frac{1}{1+i} \right)^t = \frac{C}{i} \left(1 - \frac{1}{(1+i)^n} \right). \qquad (10)$$

Again, we give a rigorous demonstration of (10) in the appendix. However, (10) can also be demonstrated very intuitively. It is easy to see that an annuity is simply a difference between two perpetuities. To see this, consider the following table:

Time	**1**	**2**	**…**	**n-1**	**n**	**n+1**	**n+2**	**…**
Perpetuity P1	C	C	C	C	C	C	C	C
Perpetuity P2	0	0	0	0	0	C	C	C
Annuity A = P1-P2	C	C	C	C	C	0	0	0

We can now express the payments to the annuity as:

$$Cash\ Flow(A) = Cash\ Flow(P1) - Cash\ Flow(P2)\ .$$

Hence, applying the principle of value additivity,

$$V_0(A) = V_0(P1) - V_0(P2)\ .$$

We have already established in the previous section that the value of *P1* at time zero is $V_0 = C/i$. Moreover, by the same principle the value of *P2* at time *n* is also *C/i*. Hence, the present value of *P2* is:

$$V_0(P2) = \frac{1}{(1+i)^n}\frac{C}{i}\ .$$

Note that the payments for *P2* start at time *n+1*. However, the present value of *P2* at time *n* is *C/i*, so we need to discount *C/i* over *n* periods, not *n+1* periods. Hence:

$$V_0(A) = V_0(P1) - V_0(P2) = \frac{C}{i} - \frac{1}{(1+i)^n}\frac{C}{i}$$
$$= \frac{C}{i}\left(1 - \frac{1}{(1+i)^n}\right).$$

By remembering how to express an annuity as a difference between two perpetuities, all you need to remember is the present value formula (2), and the formula for perpetuities (9), and you will always know how to derive (10) and what number to use in the exponent for *n*.

Example 15:

Star Mortgages considers buying a mortgage from Moon Bank. The mortgage was originally a thirty-year fixed rate mortgage and still has exactly twenty years of monthly payments. The mortgage rate agreed on the mortgage is 9%, and the monthly payments are $1,500 per month. How much is Star Mortgages willing to pay when they purchase the mortgage if the current 20-year mortgage rate is 6%? The first important observation here is that the original mortgage rate of 9% is completely irrelevant here. Since interest rates have fallen, Moon is discounting future payments at 6%. Hence, the periodic interest rate is i=0.06/12=0.005 or 0.5%, and the number of periods is 20*12=240 months. Hence, we use (10) as follows:

$$V_0 = \frac{\$1{,}500}{0.005}\left(1 - \frac{1}{1.005^{240}}\right) = \$209{,}371$$

Hence, Star is willing to pay $209,371 for this mortgage.

If you take out a mortgage you are probably more interested in a different question: given the interest rate and the amount you wish to borrow, what is your monthly repayment? Note that (10) has a very simply structure. You simply multiply the constant periodic payment C by a factor that only depends on the interest rate and the number of periods n. This factor is called the **annuity factor**. We use the symbol $A_n(i)$ for this factor, that is defined as:

$$A_n(i) = \frac{1}{i}\left(1 - \frac{1}{(1+i)^n}\right) \tag{11}$$

We can therefore express (10) as:

$$V_0 = C * A_n(i)$$

Now it is simple to solve for C:

$$C = \frac{V_0}{A_n(i)} \tag{12}$$

Equation (12) has an important interpretation. Suppose we wish to take out a loan with an amount V_0, then C is the constant periodic payment we need to make in order to repay the loan over n periods if the periodic interest rate is i.

Example 16:

You want to take out a mortgage of \$200,000 on your house, and you are offered an interest rate (APR) of 6% on a 15-year mortgage. Interest is compounded monthly. What is your monthly repayment? Simply apply (11) and (12) above with i=6%/12=0.005% and n=15*12=180 to get:

$$C = \frac{\$200{,}000}{\frac{1}{0.005}\left(1 - \frac{1}{1.005^{180}}\right)} = \$1{,}687.71$$

Note that this is a the gross payment for interest and principal, and does not take into account tax deductions or escrow payments connected with this loan.

It is instructive to rewrite (12) by substituting for the annuity factor:

$$C = \frac{V_0 * i}{1 - \left(\frac{1}{1+i}\right)^n}$$

The numerator of this expression is already familiar from our discussion of perpetuities. If we do not make any repayments of principal, then we must make monthly interest payments equal to $V_0 * i$. Then we would owe the full amount (or **par amount**) of the loan at the end of the period. The expression in the denominator is clearly smaller than one, hence it increases the payment *C* to account for the fact that we make repayments of principal as well as interest payments.

Example 14 (cont.)

In example 14 we already established that the monthly payment has to exceed \$750. We can now calculate it as:

$$C = \frac{\$150{,}000}{\frac{1}{0.005}\left(1 - \frac{1}{1.005^{360}}\right)} = \$899.33$$

Note, however, that the composition of the monthly payment *C* between interest and principal changes over the lifetime of the loan. In the early stages of repaying a loan, interest accounts for most of the monthly payments. However, as you repay the principal, the loan balance decreases and so does the interest component, and at the end the monthly payments are almost entirely repayments of principal. The mechanics of this are the subject of a repayment schedule, which is best demonstrated by way of an example.

Example 17:

Reconsider example 16 and consider the first monthly payment of $1,687.71. After one month, the interest owed on $200,000 is exactly $1,000, or 0.5% of $200,000. Hence, the remaining $687.71 is a repayment of principal, and at the end of the first month, you owe the bank the remaining

$200,000-$687.71=$199,312.29 .

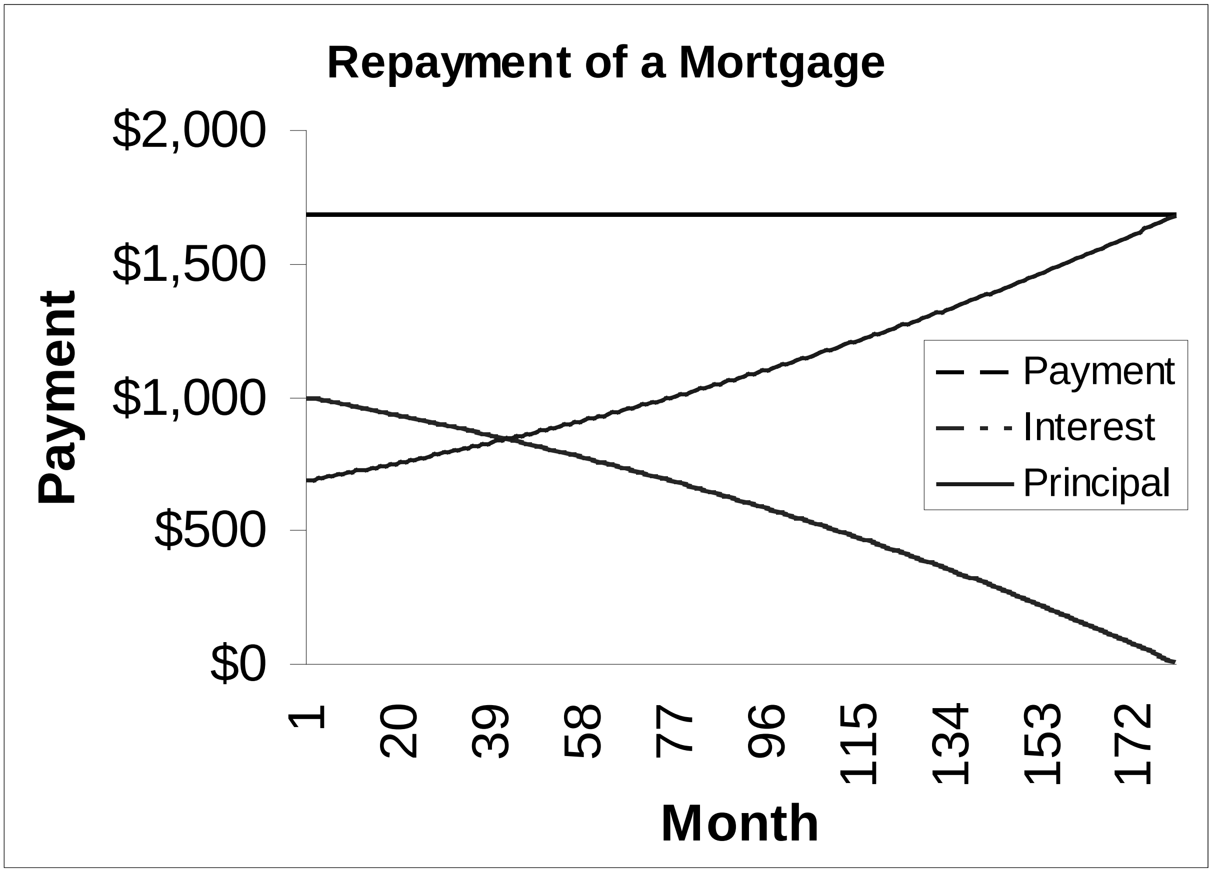
Hence, for the second month you have to pay interest only on this slightly reduced amount, which is $996.56, so that your principal payment in the second month must be

$1687.71-$996.56=$691.15 .

The following table shows the first and last few months of the repayment schedule:[8]

Month	Initial Balance	Payment	Interest	Principal	Ending Balance
1	$200,000.00	$1,687.71	1,000.00	$687.71	$199,312.29
2	$199,312.29	$1,687.71	996.56	$691.15	$198,621.13
3	$198,621.13	$1,687.71	993.11	$694.61	$197,926.53
4	$197,926.53	$1,687.71	989.63	$698.08	$197,228.45
…	…	…	…	…	…
178	$5,012.93	$1,687.71	25.06	$1,662.65	$3,350.28
179	$3,350.28	$1,687.71	16.75	$1,670.96	$1,679.32
180	$1,679.32	$1,687.71	8.40	$1,679.32	$0.00

The figure displayes the time-series patterns of the total payment, and its decomposition into principal and interest.



So far we have analyzed the present value of annuities. Often we are also interested in the future value of an annuity, for example, if you want to determine the value you accumulate in a pension plan if you make constant payments over a certain period of time. Using (10) and applying (1) gives:

$$V_n = (1+i)^n * \frac{C}{i}\left(1 - \frac{1}{(1+i)^n}\right) = \frac{C}{i}\left((1+i)^n - 1\right). \qquad (13)$$

Example 18:

Today is your 35th birthday, and you reckon you can put aside $2,400 a quarter into a pension plan where your money accumulates at a rate of 5% p. a.,

---

8 See the spreadsheet for the calculation of the complete repayment schedule.

compounded quarterly. How much will you have accumulated in the plan after you made the last payment on your 65th birthday? Effectively, you make 30*4=120 quarterly payments of $2,400 over 30 years that are compounded at 1.25% per quarter. Using (13):

$$V_{120} = \frac{\$2,400}{0.0125}\left(1.0125^{120} - 1\right) = \$660,520.94 \ .$$

1.7. Growing Perpetuities and Growing Annuities

Our analysis of perpetuities and annuities above was limited by the assumption that payments stay constant over time. It is actually straightforward to analyze a more general case, where payments are allowed to grow at a constant rate g over time. Hence, we postulate:

$$\begin{aligned}
C_2 &= (1+g)C_1 \\
C_3 &= (1+g)C_2 = (1+g)^2 C_1 \\
&\dots \\
C_n &= (1+g)C_{n-1} = \dots = (1+g)^{n-1} C_1
\end{aligned}$$

Example 19:

The US government wishes to issue perpetual bonds. In order to provide investors with a more attractive investment, the treasury department decides that the annual coupon on one bond is $100 in the first year, and growing at a rate of 3% per year after that. Then the annual coupon is $103 in the second year of the bond, $106.09 in the third year, and by the tenth year has become a remarkable $130.48. Evidently, the mechanics are the same as in the future value calculation from (1) above, where the growth rate g=3% takes the place of the interest rate.

It turns out that such a **growing perpetuity** is just as straightforward to value as a constant perpetuity. We can adapt (9) to give:

$$V_0 = \sum_{t=1}^{t=\infty} \frac{C_t}{(1+i)^t} = C_1 * \sum_{t=1}^{t=\infty} \frac{(1+g)^{t-1}}{(1+i)^t} = \frac{C_1}{i-g} \qquad (14)$$

Equation (14) is derived in the appendix. Note that this result is only valid if $i>g$: V_0 can never become negative if C_1 is positive.

Example 20:

Reconsider example 19, and suppose that the annual growth rate of coupons is 3%, and the interest rate is 5%. Apply (14) to give:

$$V_0 = \frac{\$100}{0.05 - 0.03} = \$5{,}000$$

Note that (9) is a special case of (14), hence we need to memorize only (14): If we set g=0 in (14) we obtain (9) again. We can apply the same logic to growing annuities, where the number of payments is finite. Similarly, a finite sequence of growing payments is **growing annuity**. The parallel expression for (10) is:

$$V_0 = \frac{C_1}{1+g} * \sum_{t=1}^{t=n} \left(\frac{1+g}{1+i}\right)^t = \frac{C}{i-g}\left(1 - \left(\frac{1+g}{1+i}\right)^n\right) \qquad (15)$$

Example 21:

Reconsider the example from your pension plan in example 18. However, suppose you expect your contributions to grow at a rate of 0.5% per quarter, so your first quarter's contributions are still going to be \$2,400 as before, but in the second quarter you now contribute \$2,412. What is the ending balance in your fund after you made the last payment on your 65th birthday now? What contribution would you have to start with if you wanted to accumulate \$1,000,000 by your 65th birthday? We compute this in two steps. The first step computes the present value as of your 35th birthday. This is from (15):

$$V_0 = \frac{\$2,400}{0.0125 - 0.005}\left(1 - \left(\frac{1.005}{1.0125}\right)^{120}\right) = \$188,878.60$$

The second step is to convert the present value into a future value using (1), which gives us $\$188,878.60*1.0125^{120}$=$838,661.28.

In order to accumulate $1,000,000 you need to increase your payments by a factor of $1,000,000/$838.661.28=1.192, which translates into a contribution in the initial quarter of $2,861.70. Note, however, that you are committed to increase this by 0.5% per year, so the finial quarter's contribution is going to be $5,180.67, almost double of what you contribute today!

Conclusion

The main purpose of this note is to illustrate discounting techniques and their applications. These techniques are fundamental for all financial calculations in subsequent lectures. It is easy to get lost in the maze of different formulas. It is often easier to memorize these by understanding the relationships between the main concepts.

- Present values and future values are basically flip sides of the same coin, simply reverse the direction in time. You only need to remember the generic formula (1) and how it relates to (2). All other future value formulas are special applications (so (4) follows from (3), (13) follows from (10)).

- The generic present value formula is (3). It encompasses all subsequent formulas as special cases. The main special cases are perpetuities and annuities.

- Think about discount rates as applying to periods. Periods are typically shorter than one year.

- Constant annuities and constant perpetuities are special cases of growing annuities and growing perpetuities respectively: Only learn (14) and (15) which are more general. (9) is a special case of (14), (10) is a special case of (15).

Hence, learning (1), (2), (3), (14) and (15) is sufficient, and everything else falls into place.

Appendix

Derivation of equation (4):

Combine (1) and (3) as follows:

$$V_n \overset{from\,(1)}{=} V_0 * (1+i)^n \overset{from\,(3)}{=} (1+i)^n * \sum_{t=0}^{t=n} \frac{C_t}{(1+i)^t}$$

$$= (1+i)^n * C_0 + (1+i)^n * \frac{C_1}{(1+i)} + ... + (1+i)^n * \frac{C_1}{(1+i)^n}$$

$$= (1+i)^n * C_0 + (1+i)^{n-1} * C_1 + ... + (1+i) * C_{n-1} + C_n$$

$$\overset{gives\,(4)}{=} \sum_{t=0}^{t=n} C_t (1+i)^{n-t}$$

Derivation of equation (7):

Dividing both sides of equation (6) by V_0 we get:

$$(1+r)^t = \left(1+\frac{R}{m}\right)^{m*t}$$

Now take the t-th root on both sides. (Apply the exponent 1/t on both sides, remembering

that $(x^t)^{1/t} = x$):

$$(1+r) = \left(1+\frac{R}{m}\right)^m$$

Subtracting 1 on both sides gives (7).

Derivation of equation (9):

We make the following substitution:

$$x = \frac{1}{1+i}$$

and observe that $0 < x < 1$. Then we can rewrite the first equality in (9) as:

$$V_0 = C * \sum_{t=1}^{t=\infty} x^t = C * \left(x + x^2 + x^3 + ...\right)$$

Multiply both sides by *1-x* to obtain:

$$\begin{aligned} V_0 * (1-x) &= C * (1-x)\left(x + x^2 + x^3 + x^4 ...\right) \\ &= C * \left(\underbrace{\{x - x^2\}}_{=x(1-x)} + \underbrace{\{x^2 - x^3\}}_{=x^2(1-x)} + ... + \underbrace{\{x^{n-1} - x^n\}}_{=x^{n-1}(1-x)} + ... \right) \\ &= C * x \end{aligned}$$

where the last equality obtains from canceling equivalent expressions and observing that for $x<1$, x^n converges to zero as n becomes large. Finally, solving for V_0 and substituting for x gives:

$$V_0 = \frac{x}{1-x} C = \frac{\frac{1}{1+i}}{1 - \frac{1}{1+i}} C = \frac{C}{i}$$

which proves (9).

Derivation of equation (10):

We use exactly the same strategy as in the derivation of equation (9). Firstly:

$$V_0 = C * \sum_{t=1}^{t=n} x^t = C * \left(x + x^2 + x^3 + ... + x^n\right)$$

then, after multiplying with 1-x:

$$
\begin{aligned}
V_0 * (1-x) &= C * (1-x)\left(x + x^2 + x^3 + x^4 \ldots + x^n\right) \\
&= C * \left(\underbrace{\left\{x - x^2\right\}}_{=x(1-x)} + \underbrace{\left\{x^2 - x^3\right\}}_{=x^2(1-x)} + \ldots + \underbrace{\left\{x^{n-1} - x^n\right\}}_{=x^{n-1}(1-x)} + \underbrace{\left\{x^n - x^{n+1}\right\}}_{=x^n(1-x)} \right) \\
&= C * \left(x - x^{n+1}\right) \\
&= C * x * \left(1 - x^n\right)
\end{aligned}
$$

Then we solve for V_0 as before:

$$
V_0 = C * \frac{x}{1-x} * \left(1 - x^n\right) = C * \frac{1}{i} * \left(1 - \left(\frac{1}{1+i}\right)^n\right)
$$

after substituting for *x*, which gives (10).

Derivation of equation (14):

We proceed as in the derivation of equation (9) above, except that we now define:

$$
x = \frac{1+g}{1+i}
$$

Then we can rewrite the left hand side of (14) as:

$$
V_0 = \frac{C_1}{1+g} \sum_{t=1}^{t=\infty} \left(\frac{1+g}{1+i}\right)^t = \frac{C_1}{1+g} \sum_{t=1}^{t=\infty} x^t
$$

Then, using parallel steps to the derivation of (9):

$$
V_0 = \frac{x}{1-x} \frac{C_1}{1+g} = \frac{\frac{1+g}{1+i}}{1 - \frac{1+g}{1+i}} \frac{C_1}{1+g} = \frac{C_1}{i-g}
$$

Note that the sum $\sum_{t=1}^{t=\infty} x^t$ converges only if x<1, which is equivalent to $g<i$. Otherwise

we would have that the sum diverges to infinity.

Derivation of equation (15):

Using the same procedure as for equations (1) and (14), we obtain:

$$V_0 = \frac{C_1}{1+g} * \sum_{t=1}^{t=n} \left(\frac{1+g}{1+i} \right)^t = \frac{C_1}{1+g} * \sum_{t=1}^{t=n} x^t$$

$$= \frac{C_1}{1+g} \frac{x}{1-x} \left(1 - x^n\right)$$

$$= \frac{C_1}{1+g} \frac{\frac{1+g}{1+i}}{1 - \frac{1+g}{1+i}} \left(1 - x^n\right)$$

$$= \frac{C}{i-g} \left(1 - \left(\frac{1+g}{1+i} \right)^n \right)$$

Important Terminology

annuity	20	future value	2
annuity factor	23	growing annuity	28
APR	14	growing perpetuity	27
compound interest	14	nominal interest rate	14
Consols	18	par amount	24
discount factor	8	present value	5
effective interest rate	14	value additivity	7

Important Formulae

Future value:

$$V_n = V_0 (1+i)^n \qquad (1)$$

Present value:

$$V_0 = \frac{V_n}{(1+i)^n} \qquad (2)$$

$$V_0 = C_0 + \frac{C_1}{(1+i)} + \frac{C_1}{(1+i)^2} + ... + \frac{C_1}{(1+i)^{n-1}} + \frac{C_1}{(1+i)^n} = \sum_{t=1}^{t=n} \frac{C_t}{(1+i)^t} \qquad (3)$$

Discount factor:

$$= \frac{1}{(1+i)^t}$$

Annuity:

$$V_0(A) = \frac{C}{i}\left(1 - \frac{1}{(1+i)^n}\right) \qquad (10)$$

Growing Perpetuity:

$$V_0 = \frac{C_1}{i-g} \qquad (14)$$

Growing annuity:

$$V_0 = \frac{C}{i-g}\left(1 - \left(\frac{1+g}{1+i}\right)^n\right) \qquad (15)$$