CUBS MSc Finance 1999-2000

ASSET MANAGEMENT: LECTURE 3

Dr. D.N. Tambakis

THE CAPITAL ASSET PRICING MODEL

Textbook: EG13 (14-15), BKM8, GT5

Readings: "Tales from the FAR Side" (Economics Focus), *The Economist*, 16.11.96

Ben-Horim, M. and H. Levy, "Total Risk, Diversifiable Risk and Nondiversifiable Risk: A Pedagogic Note", Journal of Financial and Quantitative Analysis, 15(2), 6/1980, pp.289-297

3.1 Overview

The capital asset pricing model (**CAPM**) is a general equilibrium model: it describes the relationship between assets' expected return and risk when all markets are in equilibrium. In other words, each investor holds an optimal portfolio, and the aggregate portfolio of all investors is the market portfolio. We have already established the basic framework for modeling risk and return in mean-standard deviation space (ER,σ). In moving from one risky asset to the whole market, we introduce the notion of market, or systematic, risk, i.e. the covariance between a single asset or portfolio and the market portfolio itself.

The **market portfolio**, denoted ***M***, is defined as the portfolio of all risky assets, where the weight on each asset is simply the market value of that asset divided by the market value of all risky assets. Thus the market portfolio is a market value-weighted average of all risky assets. Since the weight on each asset is equal to its percentage share of the total market value, the sum of all weights is 1. Approximations to the market portfolio include the FTSE-All Share index in the UK and the S&P's 500 index in the US, whose weights are calculated by market capitalization. We shall see that, under the restrictive set of assumptions underlying the CAPM, all investors will optimally hold the market portoflio and the riskless asset regardless of their risk preferences. The relative share of the two will depend on the relative risk preferences of each individual investor.

3.2 Portfolio Risk and Market Risk: Introducing Beta

In order to evaluate asset *i*'s contribution to the risk of a portfolio *P*, we divide the covariance between *i* and *P* by the overall variance of the portfolio: σ_{iP} / σ^2_P. This gives us a normalized indication of asset *i*'s share of total portfolio risk. Reasoning analogously, the contribution of asset *i* to the risk of the market portfolio is measured by the same ratio and is defined as the **beta** (β) of the asset:

$$\beta_i = \frac{\sigma_{im}}{\sigma_m^2} \tag{1}$$

The denominator is now the variance of the market portfolio, computed for example using monthly data over the relevant time period. By definition, the beta of the market portolio itself is 1: the contribution of the market portfolio to its own risk is 100 percent. At the opposite extreme, the beta of an asset which is independent of the market (e.g. a riskless asset) is zero: as the asset's covariance with the market is zero, it has no contribution to market risk. The riskless asset's standard deviation is zero, so its returns are trivially unaffected by changes in market conditions.

More generally, the standard deviation of an asset whose beta is larger (smaller) than 1 changes more (less) than proportionately in reaction to changes in market conditions. Thus, an asset whose beta is greater (less) than 1 has a relatively greater (smaller) contribution to the risk of a portfolio. For example, a beta of 1.5 implies that a 10% market standard deviation leads to an amplified 15% standard deviation in the asset's return, while a beta of 0.5 implies that a 10% market standard deviation only leads to a 5% deviation in the asset's return.

Given knowledge of the market portfolio and its overall risk, we may want to compute the betas of assets or portfolios. In fact, the definition of beta in equation (1) can be derived using our distinction between an asset's or portfolio's *unique* (**unsystematic**) risk and its *market* (**systematic**) risk from lecture 2. We then showed that whereas the portfolio's unsystematic risk can be eliminated by diversification, its market risk cannot. This allows for a powerful tool for analyzing the portfolio's risk, and hence for assessing its expected perormance. For simplicity we consider the case of a single asset (the **single-index** model), whose return is decomposed as follows:

$$R_i = \alpha_i + \beta_i R_m \tag{2}$$

The constant α_i captures the influence on the return of all factors unique to the asset, and β_i captures returns' sensitivity to market conditions. The first term summarizes the asset's unsystematic risk, while the second its systematic risk, that is the degree to which its performance is affected by market movements. In turn, unsystematic risk itself may be decomposed into a constant term c_i and an error term e_i, which is normally distributed with expected value zero and variance σ_e^2. The constant captures the influence on returns of all unsystematic and predictable factors (for example, unique to the asset's sector in the stock market), while the error term captures all unsystematic unpredictable factors. For simplicity, the error term's distribution is assumed to be the same for all assets.Since the error term and the market are independent, their covariance is zero: $\sigma_{em} = E[\, e_i(R_m - ER_m)]=0$. Equation (2) can then be written as:

$$R_i = c_i + \beta_i R_m + e_i \tag{3}$$

Taking expectations in (3) yields:

$$ER_i = c_i + \beta_i ER_m \tag{4}$$

so the asset's expected return is just the sum of its unsystematic expected return c_i and the systematic influence of the market's expected return, weighted by its beta coefficient β_i.

The asset's variance is also decomposed along the same systematic and unsystematic. components. Recalling that the covariance between the error term and the market's return is zero by definition, the variance is given by:

$$\sigma_i^2 = \beta_i^2 \sigma_m^2 + \sigma_e^2 \tag{5}$$

Note that the unsystematic expected returns c_i does not enter into (5) because it is a constant. Rather, the asset's risk is completely determined by its unsystematic variance and by its sensitivity to market variance. Note also that, in contrast to expected returns, the sensitivity of the asset's risk to market risk is now given by the square of the beta coefficient.

Now consider the case of two assets *i* and *j*. Substituting their returns and expected returns definitions from above yields their covariance to be:

$$s_{ij} = b_i b_j s_M^2 \tag{6}$$

The assets' covariance is proportional to their betas and to the market portfolio's variance. It is easy to check that if both assets have a non-zero beta they cannot be independent. Thus each asset's market risk provides a yardstick for measuring their covariance. The step from here to deriving each asset's beta is immediate: if one of the two assets is the market portfolio itself, then its beta equals 1. Substituting $j=M$ in equation (6) yields asset *i*'s beta to be:

$$b_i = \frac{s_{im}}{s_m^2} \tag{7}$$

which is identical to the definition of beta in equation (1). Notice that all of the preceeding discussion could alternatively have been framed in terms of a portfolio *P*'s beta. However, if all *i* (i=1,...,N) are single assets, then summing all of their returns as defined in equation (2) and weighting each term in the sum by x_i (each asset's weight in the portfolio) implies that the portfolio's beta is given by:

$$b_P = \sum_i x_i b_i \tag{8}$$

The linear relationship between the betas of the underlying assets and the similar linear relationship for the portfolio's expected return allow us to express the variation of expected return as market risk changes on a straight line. This result is very useful in practice.

3.3 <u>Assumptions Underlying the Standard CAPM</u>

Before describing the relationship between risk and expected return underlying the standard CAPM, we briefly discuss the necessary restrictions to the set of possibilities open to the investor. For the transition from a world of one investor to the general equilibrium setting of the CAPM several strong assumptions are required. The most important are: (a) the absence of transaction costs, so expected return is only related to risk, (b) perfect competition, so an individual investor's decisions have no effect on prices, and (c) homogeneous expectations, so that all investors form the same assessement of assets' market risk, and hence end up desiring the same optimal portfolio.

Another important assumption which is often subsumed in practice is that investors only care about assets' risk and expected return. As we have already argued, this presupposes that the distributions of assets' returns are (log)normal, a claim which may not be true for particular assets. However, the wide-spread adoption of the CAPM in financial practice suggests that the normal distribution may be a good approximation for most purposes.

In addition to these key assumptions, the standard CAPM involves others, such as the absence of lending and borrowing restrictions on a riskless asset, the absence of a personal income tax, and infinite divisibility and full marketability of all assets, including human capital. Although each of these plays a particular role in simplifying the analysis, they can be relatively easily relaxed and/or generalized. The great number of extensions to the standard CAPM in the financial literature reflect the desire to generalize the original setup to be able to analyze more realistic situations.

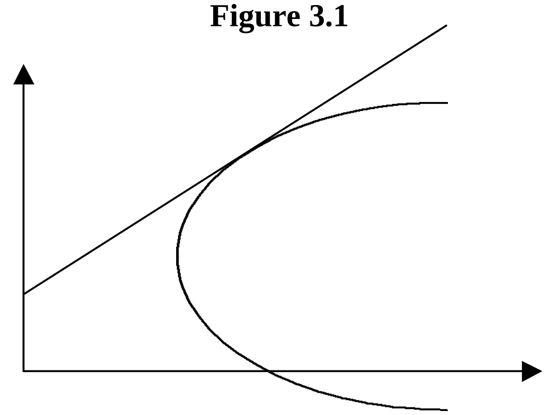
3.4 <u>The Standard Version of the CAPM</u>

Recall from lecture 2 (equation (12)), that for any portfolio E lying on the efficient frontier, with riskless lending and borrowing a straight line can can be constructed connecting it and the riskless asset. This relationship is often known as the **capital market line**:

$$ER_P = R_F + \frac{ER_M - R_F}{s_M} s_P \tag{9}$$

The standard deviation of the efficient portfolio measures its risk, and the fraction multiplying it may be thought of as the *price* of this risk, i.e. the extra expected return investors must receive for each extra unit of risk. Such a line is depicted on the following diagram:

Figure 3.1



With the restrictive assumptions of homogeneous expectations and riskless lending and borrowing at the same constant rate, all investors will hold the same risky portfolio on the efficient frontier. In other words, although their share of that portfolio may vary depending upon their relative risk aversion, the portfolio itself is common to all investors. In equilibrium, that portfolio has to be the market value-weighted portfolio of all risky assets, which we have defined as the market portfolio. Since the weight on each asset in each investor's portfolio will be its proportion of total market value, the asset weights must sum to one for each investor. The only difference between portfolios is one of scaling, as different investors may have different wealth.

The property that rational investors following the assumptions of the CAPM will optimally hold only two assets (portfolios), the riskless asset and the market portfolio, is known as the two mutual fund (unit trust) theorem, or simply as the *separation theorem*. Although a formal proof of this result is beyond the scope of this course, its essence should be intuitive from our purposes. In particular, it should be clear that the market portfolio itself belongs to the efficient frontier, and as such can be combined with the riskless asset.

First, the market portfolio is a minimum-variance portfolio, as it is a linear combination (with market-valued weights) of all investors' risky asset portfolios, each of which is on the minimum-variance frontier. Second, since each investor's optimal risky asset portfolio is efficient, the market portfolio cannot lie on the lower part of the minimum variance frontier. For, if it did, its expected return would be a linear combination of the expected returns of risky asset portfolios which would not lie on the efficient frontier. Therefore, the market portfolio lies on the efficient frontier.

Returning to the relationship between market risk and expected return, the expected return on any portfolio is the weighted sum of its constituent assets' expected returns:

$$ER_P = \sum_i x_i ER_i \tag{10}$$

and from equation (8) we know that the beta of the portfolio is just the weighted sum of the betas of its constituent assets. To arrive at the CAPM, consider the case where the two assets in question are the riskless asset, with expected return R_F, and the market value-weighted portfolio of all risky assets, i.e. the market portfolio *M*. Assume that the representative investor places weight *x* of their wealth on the riskless asset and the remainder (1-*x*) on the market portfolio. The expected return and risk of combinations of the riskless asset and the market portfolio then are:

$$\begin{aligned} ER_P &= xR_F + (1-x)ER_M \\ \beta_P &= x\beta_F + (1-x)\beta_M \end{aligned} \tag{11}$$

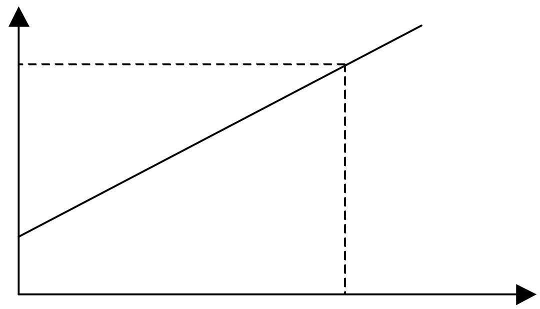
However, we already know that the betas of the riskless asset and the market portfolio are 0 and 1. So the beta of the portfolio is just (1-*x*), the weight placed on the market portfolio. Substituting portfolio *P*'s beta into the expression for its expected return yields:

$$\begin{aligned} ER_P &= R_F + \beta_P (ER_M - R_F) \Leftrightarrow \\ ER_P &= R_F + \frac{\sigma_{PM}}{\sigma_M^2}(ER_M - R_F) \end{aligned} \tag{12}$$

With riskless lending and borrowing, the portfolio's characteristics in (ER,β) space thus define a linear relationship known as the **security market line**.

Note that portfolios such as *A* or *B* describe different investors' risk preferences, while portfolios such as *C* or *D* which lie off the line are not sustainable in equilibrium:

Figure 3.2



Equation (12), determining portfolio *P*'s equilibrium expected return as a function of its market risk is known as the **standard CAPM**. Expressed in terms of asset prices, it allows us to price assets in equilibrium based on their market risk. Notice that the CAPM relation says that investors are *only* compensated for bearing market (systematic) risk, given by the magnitude of σ_{PM}/σ_M. As discussed earlier, the price of that risk—also known as the **Sharpe ratio**—is then given by $(ER_M\text{-}R_F) / \sigma_M$. Referring to Figure 3.1, a rational investor maximizing the slope of the capital market line connecting the riskless asset to portfolios on the efficient frontier is equivalently maximizing the price of extra market risk. In equilibrium, the CAPM's assumptions imply that, for al investors, the relevant optimal portfolio on the efficient frontier is the market portfolio.

The standard CAPM thus says nothing about the asset's unique, or unsystematic, risk. In other words, all diversifiable (unsystematic) portfolio risk has been eliminated as all investors optimally hold an equal share of the market portfolio of all risky assets. In its standard form, the CAPM states that the only cause of change in expected returns, other than the market's and the riskless asset's returns, is the portfolio's beta. Empirical support for this strong proposition over the past 40 years has been mixed, thus prompting a number of extensions of the standard model.

3.5 Extensions to the Standard CAPM

The simplest extension to the standard CAPM involves dropping the assumption of no lending/borrowing constraints while maintaining the assumption of short sales. In reality, although lending funds is free at the riskless rate, borrowing is not, or if it is allowed it involves a higher borrowing rate. It turns out that such an extension does not affect the qualitative features of the model. Indeed, it offers a simple analytical derivation of equation (12) which links with our general discussion on risk and expected return in lecture 2.

We concentrate on the case where there is no riskless rate of interest, so neither lending nor borrowing are allowed. It then still follows from equations (8) and (10) that different portfolio combinations will all lie on the straight line which we have called the capital market line. In particular, the market portfolio will also lie on this line as it is a linear combination of all individual risky assets. We may, therefore, select a portfolio *Z* with zero beta lying on the vertical axis and the market portfolio *M* as two points which together specify the straight line.

Note that although a riskless asset uncorrelated with the market portfolio (zero-beta) does not exist under our assumptions, we can always find a risky portfolio uncorrelated with the market which lies on the minimum variance frontier by extending the horizontal line corresponding to the riskless asset's expected return. The straight line linking *Z* and *M* then becomes the security market line, and the resulting version of the CAPM is commonly known as the **zero-beta**, or **two-factor CAPM**, in reference to the fact that all portfolios are formed as combinations of two portfolios (**factors**), the zero-beta portfolio and the market portfolio.

The optimal portfolio choice for each investor results from a similar exercise as in the case of two assets or portfolios in lecture 2. Recall that we had equation (2.16) relating the weighted sum of the variance and covariances of asset i with all other assets (scaled by a parameter λ) and the asset's expected return over the riskless rate of return. The riskless rate now becomes the expected return of the zero-beta portfolio *Z*:

$$l x_i s_i^2 + l \sum_{j \neq i} x_j s_{ij} = ER_i - ER_Z, i = 1, \ldots, N \qquad (13)$$

This is a system of N equations, one for each risky asset. The LHS of (13) is just the covariance of asset i with the market, so we write:

$$\lambda \sigma_{iM} = ER_i - ER_Z, i = 1,\ldots,N \tag{14}$$

which can be expressed as:

$$ER_i = ER_Z + \lambda \sigma_{iM\,Z}, i = 1,\ldots,N \tag{15}$$

Since equation (15) holds for every asset, it also holds for the market portfolio, which is a linear combination of all assets. Substituting i=M gives us the coefficient λ as the ratio of the excess market expected return over the zero-beta portfolio and the market variance:

$$\lambda = \frac{ER_M - ER_Z}{\sigma_M^2} \tag{16}$$

Substituting this expression for λ back into equation (15) yields the equilibrium relationship between risk and expected return for any asset for the zero-beta model:

$$\begin{aligned} ER_i &= ER_Z + \frac{\sigma_{iM}}{\sigma_M^2}(ER_M - ER_Z) \\ ER_i &= ER_Z + \beta_i (ER_M - ER_Z), i = 1,\ldots,N \end{aligned} \tag{17}$$

So the standard CAPM relationship between market risk expected return is maintained in the absence of a riskless asset. As argued above, there is an unlimited number of potential zero-beta portfolios offering expected return ER_Z. Rational investors will choose the combination of Z and M lying on the minimum variance frontier in (ER,σ) space. It is easy to check that the minimum-variance zero-beta portfolio cannot be on the efficient frontier: on the one hand, it is not the global minimum variance portfolio, and on the other hand, linear combinations of Z and the market portfolio offer higher expected return than Z itself. However, the zero-beta CAPM shows that all investors optimize by holding some combination of *Z* and *M*. Since the aggregate portfolio is the market portfolio, the aggregate holding of *Z* must be zero (long positions must net out short positions).

A multitude of extensions of the standard CAPM in recent decades have covered a variety of financial specifications. Examples include allowing for personal taxes and/or differential lending and borrowing rates, introducing non-zero transaction costs in a multiperiod dynamic specification (the standard CAPM works period after period), modeling the covariance of assets with lifelong consumption rather than with the market (the consumption CAPM) etc. However, despite the differences in modeling among different alternatives, the main qualitative asset pricing implication of a linear and positive relationship between systematic risk—however we may want to define it—and assets' expected return is preserved.

3.6 Empirical Testing

As may be expected of any model with a general specification, the CAPM is difficult to test empirically. There are two principal interrelated difficulties. The first is that, in principle, expectations of the return on the market portfolio and particular assets are unobservable forward-looking variables, and as such can only be approximated by rolling averages of past returns. Moroever, underlying the averaging of observations is an assumption that returns are (log)normally distributed: averaging will then lead to biased estimates of expected returns if returns are in fact non-normally distributed. The second difficulty is that the market portfolio is itself unobserved, and any proxy chosen (such as the FTSE-100 or the S&P 500 indexes) will be a good approximation at best. Indeed, even if one had precise estimates of all traded assets' expected returns and betas, in order to have access to all risky assets under the CAPM's assumptions one should include non-marketable assets such as human capital. These are hard to quantify and their absence also induces biases in testing procedures.

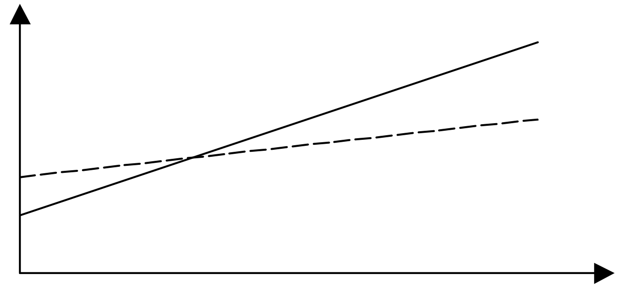
Therefore, the main equilibrium implication of the CAPM, i.e. that the market portfolio is efficient in (*ER*,σ) space, is difficult to test for. In practice, a great number of tests focus on linear regression techniques to test for the appropriate signs and magnitudes of the linear relation between expected return and market risk. Tests of the standard version would regress excess market returns on excess returns on individual assets using ordinary least squares (OLS), such as:

$$R_{it} - R_{Ft} = a_i + b_i (R_{Mt} - R_{Ft}) + e_{it} \qquad (18)$$

If the CAPM is true, we expect to find a zero value for the α intercept and a β value corresponding to the average covariance with the market portfolio of the asset or portfolio in question. Tests can be carried out using either (a) time series for a single asset, (b) cross-sectional data over a particular period for a range of assets, or (c) panel data combining the two approaches.

Early tests of the standard CAPM found strong evidence of the linear relationship in the stock market. However, later tests of the standard CAPM progressively showed that the model was unable to explain the observed variation in average stock returns by the underlying variation in betas: the slope of the straight line in (*ER*,β) slope was too low. Moreover, the estimate of the riskless rate was too large compared to that available on Treasury bills, i.e. the intercept of the straight line on the vertical was too high. In addition, the error term is assumed to be uncorrelated with the beta term, a strong assumption in the case of portfolios with a relatively small number of assets. These biases may be summarized by the following diagram:

Figure 3.3



Tests of the zero-beta version of the CAPM (with no riskless asset) appear to have been more successful. For example, Black, Jensen and Scholes (1972) write the basic linear relation from equation (17) as:

$$R_{it} = ER_Z(1 - b_i) + b_i R_{Mt} + e_{it} \qquad (19)$$

where the OLS regression on actual data is:

$$R_{it} = a_i + R_F(1 - b_i) + b_i R_{Mt} + e_{it} \qquad (20)$$

Equating the intercept terms in (19) and (20) implies:

$$a_i = (ER_Z - R_F)(1 - b_i) \qquad (21)$$

Therefore, given that the expected return on the zero-beta portfolio is larger than the riskless rate, the intercept term should be negative (positive) if the beta estimate is greater (less) than 1. Such testable predictions allowing the data to discriminate between alternative hypotheses have lent empirical support to the zero-beta CAPM.

Overall, despite the varying degrees of success of different CAPM versions against the evidence, the methodology as a whole has had a profound impact on financial practice which is likely to continue. Its influence on portfolio management cannot be overestimated. Moreover, despite being relatively abstract and simplistic, the linear relationship between expected return and market risk is widely used in firms' capital budgeting and investment decisions to estimate the expected rate of return on capital investments and management projects. The underlying valuation principles of the CAPM are applied in fields as diverse as unit trust (mutual fund) risk management and regulation and pricing schemes for public and privatized utilities.